

A New Derivation of the Picard-Fuchs Equations for Effective $N = 2$ Super Yang-Mills Theories

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Abstract

A new method to obtain the Picard–Fuchs equations of effective, $N = 2$ supersymmetric gauge theories in 4 dimensions is developed. It includes both pure super Yang–Mills and supersymmetric gauge theories with massless matter hypermultiplets. It applies to all classical gauge groups, and directly produces a decoupled set of second-order, partial differential equations satisfied by the period integrals of the Seiberg–Witten differential along the 1-cycles of the algebraic curves describing the vacuum structure of the corresponding $N = 2$ theory.

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1 Introduction

There have been enormous advances in understanding the low-energy properties of supersymmetric gauge theories in the last couple of years [1]. In particular, for $N = 2$ gauge theories with N_f matter hypermultiplets, the exact solution for the low-energy properties of the Coulomb phase of the theory is given in principle by a hyperelliptic curve [2]–[18]. In practice, however, a great deal of additional work is required to extract the physics embodied in the curve that describes the theory in question. A given theory is characterised by a number of moduli [1]–[18] which are related to the vacuum expectation values of the scalar fields of the $N = 2$ vector multiplet and the bare masses of the matter hypermultiplets. If the scalar fields of the matter hypermultiplets have vanishing expectation values, then one is in the Coulomb phase of the theory; otherwise one is in a Higgs or in a mixed phase [1, 17]. This paper will only be concerned with the Coulomb phase of asymptotically free $N = 2$ supersymmetric theories.

The Seiberg-Witten (SW) period integrals

$$\vec{\Pi} = \begin{pmatrix} \vec{a}_D \\ \vec{a} \end{pmatrix} \quad (1.1)$$

are related to the prepotential [19] $\mathcal{F}(\vec{a})$ characterising the low-energy effective Lagrangian by

$$a_D^i = \frac{\partial \mathcal{F}}{\partial a_i} \quad (1.2)$$

One can use the global monodromy properties of $\vec{\Pi}$ to essentially fix it, and then find the prepotential $\mathcal{F}(\vec{a})$ by integration. In practice, one needs to construct the SW periods $\vec{\Pi}$ from the known, auxiliary hyperelliptic curve; not an easy task for groups with rank 2 or greater. One particular way of obtaining the necessary information is to derive a set of Picard-Fuchs (PF) equations for the SW period integrals. The PF equations have been formulated for $SU(2)$ and $SU(3)$ with $N_f = 0$ [11, 20], and for $N_f \neq 0$ for massless hypermultiplets ($m = 0$) [22, 24]. The solutions to these equations have been considered for $SU(2)$ with $N_f = 0, 1, 2, 3$, for $SU(3)$ with $N_f = 0, 1, \dots, 5$, and for other classical gauge groups in [11, 22, 24, 26]. In the particular case of massless $SU(2)$, the solutions to the PF equations are given by hypergeometric functions [20, 22], while for $N_f = 0$ $SU(3)$, they are given (in certain regions of moduli space) by Appell functions, which generalise the hypergeometric

function [11]. In other regions of the $SU(3)$ moduli space, only double power-series solutions are available. Thus, even given the hyperelliptic curve characterising the Coulomb phase of an $N = 2$ supersymmetric gauge theory, considerable analysis is required to obtain the SW periods and the effective Lagrangian in various regions of moduli space.

The first task in such a programme is to obtain the PF equations for the SW period integrals. Klemm *et al.* [11] describe a particular procedure which enables them to obtain the PF equations for $SU(3)$ with $N_f = 0$, which in principle is applicable to other theories as well. One would like to obtain and solve the PF equations for a wide variety of theories in order to explore the physics contained in particular solutions, and also to obtain an understanding of the general features of $N = 2$ gauge theories. Therefore it is helpful to have an efficient method for constructing PF equations from a given hyperelliptic curve, so that one can obtain explicit solutions for groups with rank greater than or equal to 2.

It is the purpose of this paper to present a systematic method for finding the PF equations for the SW periods which is particularly convenient for symbolic computer computations, once the hyperelliptic curve appropriate to a given $N = 2$ supersymmetric gauge theory is known. Our method should be considered as an alternative to that of Klemm *et al.* [11]. A key element in our treatment is the Weyl group symmetry underlying the algebraic curve that describes the vacuum structure of the effective $N = 2$ SYM theory (with or without massless hypermultiplets). For technical reasons, we will not treat the theories with non-zero bare masses, but leave a discussion of such cases to subsequent work [31].

This paper is organised as follows. In section 2, our method is described in general, so that given a hyperelliptic curve for some $N = 2$ theory, one will obtain a coupled set of partial, first-order differential equations for the periods. The method is further elucidated in section 3, where a technique is developed to obtain a decoupled set of partial, second-order differential equations satisfied by the SW periods. A number of technical details pertaining to the application of our method to different gauge groups (both classical and exceptional) are also given in section 3. Some relevant examples in rank 2 are worked out in detail in section 4, for illustrative purposes. Our results are finally summarised in section 5.

Appendix A deals with a technical proof that is omitted from the body of the text. An extensive catalogue of results is presented in appendix B, including $N_f \neq 0$ theories (but always with zero bare mass). Explicit solutions to the PF equations themselves for rank

greater than 2 can be quite complicated, so we will restrict this paper to the presentation of the methods and a catalogue of PF equations. Solutions to some interesting cases will be presented in a sequel in preparation [31]. The methods of this paper will have applications to a variety of questions, and are not limited to the SW problem.

2 The Picard-Fuchs Equations: Generalities

2.1 Formulation of the problem

Let us consider the complex algebraic curve

$$y^2 = p^2(x) - x^k \Lambda^l \quad (2.1)$$

where $p(x)$ is the polynomial

$$p(x) = \sum_{i=0}^n u_i x^i = x^n + u_{n-2} x^{n-2} + \dots + u_1 x + u_0 \quad (2.2)$$

$p(x)$ will be the characteristic polynomial corresponding to the fundamental representation of the Lie algebra of the effective $N = 2$ theory. We can therefore normalise the leading coefficient to 1, *i.e.*, $u_n = 1$. We can also take $u_{n-1} = 0$, as all semisimple Lie algebras can be represented by traceless matrices. The integers k , l and n , as well as the required coefficients u_i corresponding to various choices of a gauge group and matter content, have been determined in [2]–[10]. From dimensional analysis we have $0 \leq k < 2n$ [1]–[18]. Λ is the quantum scale of the effective $N = 2$ theory. Without loss of generality, we will set $\Lambda = 1$ for simplicity in what follows. If needed, the required powers of Λ can be reinstated by imposing the condition of homogeneity of the equations with respect to the (residual) R -symmetry.

Equation 2.1 defines a family of hyperelliptic Riemann surfaces Σ_g of genus $g = n - 1$ [28]. The moduli space of the curves 2.1 coincides with the moduli space of quantum vacua of the $N = 2$ theory under consideration. The coefficients u_i are called the *moduli* of the surface. On Σ_g there are g holomorphic 1-forms which, in the canonical representation, can be expressed as

$$x^j \frac{dx}{y}, \quad j = 0, 1, \dots, g - 1 \quad (2.3)$$

and are also called *abelian differentials of the first kind*. The following g 1-forms are meromorphic on Σ_g and have vanishing residues:

$$x^j \frac{dx}{y}, \quad j = g + 1, g + 2, \dots, 2g \quad (2.4)$$

Due to this property of having zero residues, they are also called *abelian differentials of the second kind*. Furthermore, the 1-form

$$x^g \frac{dx}{y} \quad (2.5)$$

is also meromorphic on Σ_g , but with non-zero residues. Due to this property of having non-zero residues it is also called an *abelian differential of the third kind*. Altogether, the abelian differentials $x^j dx/y$ in equations 2.3 and 2.4 will be denoted collectively by ω_j , where $j = 0, 1, \dots, 2g$, $j \neq g$. We define the *basic range* R to be $R = \{0, 1, \dots, \check{g}, \dots, 2g\}$, where a check over g means the value g is to be omitted.

In effective $N = 2$ supersymmetric gauge theories, there exists a preferred differential λ_{SW} , called the Seiberg-Witten (SW) differential, with the following property [1]: the electric and magnetic masses a_i and a_i^D entering the BPS mass formula are given by the periods of λ_{SW} along some specified closed cycles $\gamma_i, \gamma_i^D \in H_1(\Sigma_g)$, *i.e.*,

$$a_i = \oint_{\gamma_i} \lambda_{SW}, \quad a_i^D = \oint_{\gamma_i^D} \lambda_{SW} \quad (2.6)$$

The SW differential further enjoys the property that its modular derivatives $\partial \lambda_{SW} / \partial u_i$ are (linear combinations of the) holomorphic 1-forms [1]. This ensures positivity of the Kähler metric on moduli space. Specifically, for the curve given in 2.1 we have [3, 5, 10]

$$\lambda_{SW} = \left[\frac{k}{2} p(x) - x p'(x) \right] \frac{dx}{y} \quad (2.7)$$

In the presence of non-zero (bare) masses for matter hypermultiplets, the SW differential picks up a non-zero residue [1], thus causing it to be of the third kind. Furthermore, when the matter hypermultiplets are massive, the SW differential is no longer given by equation 2.7. In what follows λ_{SW} will never be of the third kind, as we are restricting ourselves to the pure SYM theory, or to theories with massless matter.

Let us define $W = y^2$, so equation 2.1 will read

$$W = p^2(x) - x^k = \sum_{i=0}^n \sum_{j=0}^n u_i u_j x^{i+j} - x^k \quad (2.8)$$

Given any differential $x^m dx/y$, with $m \geq 0$ an integer, let us define its *generalised μ -period* $\Omega_m^{(\mu)}(u_i; \gamma)$ along a fixed 1-cycle $\gamma \in H_1(\Sigma_g)$ as the line integral [27]

$$\Omega_m^{(\mu)}(u_i; \gamma) := (-1)^{\mu+1} \Gamma(\mu+1) \oint_{\gamma} \frac{x^m}{W^{\mu+1}} dx \quad (2.9)$$

In equation 2.9, $\Gamma(\mu)$ stands for Euler's gamma function, while $\gamma \in H_1(\Sigma_g)$ is any closed 1-cycle on the surface. As γ will be arbitrary but otherwise kept fixed, γ will not appear explicitly in the notation. The *usual* periods of the Riemann surface (up to an irrelevant normalisation factor) are of course obtained upon setting $\mu = -1/2$, taking $m = 0, 1, \dots, g-1$, and γ to run over a canonical (symplectic) basis of $H_1(\Sigma_g)$ [28]. However, we will find it convenient to work with an arbitrary μ which will only be set equal to $-1/2$ at the very end. The objects $\Omega_m^{(\mu)}$, and the differential equations they satisfy (called Picard-Fuchs (PF) equations), will be our prime focus of attention. With abuse of language, we will continue to call the $\Omega_m^{(\mu)}$ *periods*, with the added adjectives *of the first, second, or third kind*, if $m = 0, \dots, g-1$, $m = g+1, \dots, 2g$, or $m = g$, respectively.

2.2 The recursion relations

We now proceed to derive a set of recursion relations that will be used to set up to PF equations.

From equation 2.8 one easily finds

$$\begin{aligned} \frac{\partial W}{\partial x} &= 2nx^{2n-1} + \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} (j+l) u_j u_l x^{j+l-1} + 2 \sum_{j=0}^{n-1} (n+j) u_j x^{n+j-1} - kx^{k-1} \\ \frac{\partial W}{\partial u_i} &= 2 \sum_{j=0}^n u_j x^{i+j} \end{aligned} \quad (2.10)$$

Solve for the highest power of x in $\partial W/\partial x$, *i.e.*, x^{2n-1} , in equation 2.10, and substitute the result into 2.9 to find

$$\begin{aligned} \Omega_{m+2n-1}^{(\mu+1)} &= (-1)^{\mu+2} \Gamma(\mu+2) \oint_{\gamma} \frac{x^m}{W^{\mu+2}} \left[\frac{1}{2n} \frac{\partial W}{\partial x} \right. \\ &\quad - \frac{1}{2n} \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} (j+l) u_j u_l x^{j+l-1} - \frac{1}{n} \sum_{j=0}^{n-1} (n+j) u_j x^{n+j-1} + \frac{k}{2n} x^{k-1} \Big] = \\ &\quad - \frac{m}{2n} \Omega_{m-1}^{(\mu)} - \frac{1}{2n} \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} (j+l) u_j u_l \Omega_{m+j+l-1}^{(\mu+1)} \end{aligned}$$

$$- \frac{1}{n} \sum_{j=0}^{n-1} (n+j) u_j \Omega_{m+n+j-1}^{(\mu+1)} + \frac{k}{2n} \Omega_{m+k-1}^{(\mu+1)} \quad (2.11)$$

To obtain the last line of equation 2.11, an integration by parts has been performed and a total derivative dropped. If $m \neq 0$, shift m by one unit to obtain from equation 2.11

$$\Omega_m^{(\mu)} = \frac{1}{m+1} \left[k \Omega_{m+k}^{(\mu+1)} - 2n \Omega_{m+2n}^{(\mu+1)} - \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} (j+l) u_j u_l \Omega_{m+j+l}^{(\mu+1)} - 2 \sum_{j=0}^{n-1} (n+j) u_j \Omega_{m+n+j}^{(\mu+1)} \right] \quad (2.12)$$

Next we use equation 2.8 to compute

$$\begin{aligned} - (1+\mu) \Omega_m^{(\mu)} &= (-1)^{\mu+2} \Gamma(\mu+2) \oint_{\gamma} \frac{x^m W}{W^{\mu+2}} = \\ &= (-1)^{\mu+2} \Gamma(\mu+2) \oint_{\gamma} \frac{x^m}{W^{\mu+2}} \left[\sum_{l=0}^n \sum_{j=0}^n u_l u_j x^{l+j} - x^k \right] = \\ &= \sum_{l=0}^n \sum_{j=0}^n u_l u_j \Omega_{m+l+j}^{(\mu+1)} - \Omega_{m+k}^{(\mu+1)} \end{aligned} \quad (2.13)$$

and use this to solve for the period with the highest value of the lower index, $m+2n$, to get

$$\Omega_{m+2n}^{(\mu+1)} = \Omega_{m+k}^{(\mu+1)} - \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} u_j u_l \Omega_{m+j+l}^{(\mu+1)} - 2 \sum_{j=0}^{n-1} u_j \Omega_{m+n+j}^{(\mu+1)} - (1+\mu) \Omega_m^{(\mu)} \quad (2.14)$$

Replace $\Omega_{m+2n}^{(\mu+1)}$ in equation 2.12 with its value from 2.14 to arrive at

$$\begin{aligned} \Omega_m^{(\mu)} &= \frac{1}{m+1-2n(1+\mu)} \left[(k-2n) \Omega_{m+k}^{(\mu+1)} \right. \\ &\quad \left. + \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} (2n-j-l) u_j u_l \Omega_{m+j+l}^{(\mu+1)} + 2 \sum_{j=0}^{n-1} (n-j) u_j \Omega_{m+n+j}^{(\mu+1)} \right] \end{aligned} \quad (2.15)$$

Finally, take $\Omega_m^{(\mu)}$ as given in equation 2.15 and substitute it into 2.14 to obtain an equation involving $\mu+1$ on both sides. After shifting $m \rightarrow m-2n$, one gets

$$\begin{aligned} \Omega_m^{(\mu+1)} &= \frac{1}{m+1-2n(\mu+2)} \left[(m-2n+1-k(1+\mu)) \Omega_{m+k-2n}^{(\mu+1)} \right. \\ &\quad + 2 \sum_{j=0}^{n-1} ((1+\mu)(n+j) - (m-2n+1)) u_j \Omega_{m-n+j}^{(\mu+1)} \\ &\quad \left. + \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} ((j+l)(1+\mu) - (m-2n+1)) u_j u_l \Omega_{m+j+l-2n}^{(\mu+1)} \right] \end{aligned} \quad (2.16)$$

We now set $\mu = -1/2$ and collect the two recursion relations 2.15 and 2.16

$$\begin{aligned} \Omega_m^{(-1/2)} &= \frac{1}{m-(n-1)} \left[(k-2n) \Omega_{m+k}^{(+1/2)} \right. \\ &\quad \left. + \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} (2n-j-l) u_j u_l \Omega_{m+j+l}^{(+1/2)} + 2 \sum_{j=0}^{n-1} (n-j) u_j \Omega_{m+n+j}^{(+1/2)} \right] \end{aligned} \quad (2.17)$$

and

$$\begin{aligned}
\Omega_m^{(+1/2)} &= \frac{1}{m+1-3n} \left[(m-2n+1 - \frac{k}{2}) \Omega_{m+k-2n}^{(+1/2)} \right. \\
&+ \sum_{j=0}^{n-1} (n+j-2(m-2n+1)) u_j \Omega_{m-n+j}^{(+1/2)} \\
&+ \left. \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} (\frac{1}{2}(j+l) - (m-2n+1)) u_j u_l \Omega_{m+j+l-2n}^{(+1/2)} \right] \quad (2.18)
\end{aligned}$$

Let us now pause to explain the significance of equations 2.17 and 2.18. The existence of a particular symmetry of the curve under consideration may simplify the analysis of these equations. For the sake of clarity, we will for the moment assume that equation 2.17 will not have to be evaluated at $m = g = n - 1$, where it blows up, and that its right-hand side will not contain occurrences of the corresponding period $\Omega_{n-1}^{(+1/2)}$. A similar assumption will be made regarding equation 2.18, *i.e.*, it will not have to be evaluated at $m = 3n - 1$, and its right-hand side will not contain the period $\Omega_{n-1}^{(+1/2)}$. In other words, we are for the moment assuming that we can restrict ourselves to the subspace of differentials ω_j where $j \in R$ (or likewise for the corresponding periods $\Omega_j^{(\pm 1/2)}$). This is the subspace of differentials of the first and second kinds, *i.e.*, the 1-forms with vanishing residue. For a given curve, the particular subspace of differentials that one has to restrict to depends on the corresponding gauge group; this will be explained in section 3, where these issues are dealt with. For the sake of the present discussion, the particular subspace of differentials that we are restricting to only serves an illustrative purpose.

Under such assumptions, equation 2.17 expresses $\Omega_m^{(-1/2)}$ as a linear combination, with u_i -dependent coefficients, of periods $\Omega_l^{(+1/2)}$. As m runs over R , the linear combination in the right-hand side of equation 2.17 contains increasing values of l , which will eventually lie outside R . We can bring them back into R by means of equation 2.18 it is a recursion relation expressing $\Omega_l^{(+1/2)}$ as a linear combination (with u_i -dependent coefficients) of $\Omega_l^{(+1/2)}$ with lower values of the subindex. Repeated application of equations 2.17 and 2.18 will eventually allow one to express $\Omega_m^{(-1/2)}$, where $m \in R$, as a linear combination of $\Omega_l^{(+1/2)}$, with $l \in R$. The coefficients entering those linear combinations will be some polynomials in the moduli u_i , in principle computable using the above recursion relations. Let us call $M^{(-1/2)}$ the matrix of such coefficients [27]. Suppressing lower indices for simplicity, we have

$$\Omega^{(-1/2)} = M^{(-1/2)} \cdot \Omega^{(+1/2)} \quad (2.19)$$

where the $\Omega_l^{(\pm 1/2)}$, $l \in R$, have been arranged as a column vector. We will from now on omit the superscript $(-1/2)$ from $M^{(-1/2)}$, with the understanding that the value $\mu = -1/2$ has been fixed.

2.3 Derivation of the Picard-Fuchs equations

Having derived the necessary recursion relations, we can now start taking modular derivatives of the periods. From equation 2.10 and the definition of the periods 2.9 we have

$$\frac{\partial \Omega_m^{(\mu)}}{\partial u_i} = (-1)^{\mu+2} \Gamma(\mu+2) \oint_{\gamma} \frac{x^m}{W^{\mu+2}} \frac{\partial W}{\partial u_i} = 2 \sum_{j=0}^n u_j \Omega_{m+i+j}^{(\mu+1)} \quad (2.20)$$

Again, the right-hand side of equation 2.20 will eventually contain values of the lower index outside the basic range R , but use of the recursion relations above will reduce it to a linear combination of periods $\Omega_l^{(\mu+1)}$ with $l \in R$. The coefficients will be polynomials in the moduli u_i ; let $D(u_i)$ be this matrix of coefficients. Setting $\mu = -1/2$, we end up with a system of equations which, in matrix form, reads

$$\frac{\partial}{\partial u_i} \Omega^{(-1/2)} = D(u_i) \cdot \Omega^{(+1/2)} \quad (2.21)$$

As a second assumption to be justified presently, suppose for the moment that the matrix M in equation 2.19 can be inverted to solve for $\Omega^{(+1/2)}$ as a function of $\Omega^{(-1/2)}$. Substituting the result into equation 2.21, one gets

$$\frac{\partial}{\partial u_i} \Omega^{(-1/2)} = D(u_i) \cdot M^{-1} \cdot \Omega^{(-1/2)} \quad (2.22)$$

Equation 2.22 is a coupled system of first-order, partial differential equations for the periods $\Omega^{(-1/2)}$. The coefficients are rational functions of the moduli u_i , computable from a knowledge of W and the recursion relations derived above. In principle, integration of this system of equations yields the periods as functions of the moduli u_i . The particular 1-cycle $\gamma \in H_1(\Sigma_g)$ being integrated over appears in the specific choice of boundary conditions that one makes. In practice, however, the fact that the system 2.22 is coupled makes it very difficult to solve. A possible strategy is to concentrate on one particular period and try to obtain a reduced system of equations satisfied by it. Decoupling of the equations may be achieved at the cost of increasing the order of derivatives. Of course, in the framework of effective $N = 2$

SYM theories, one is especially interested in obtaining a system of equations satisfied by the periods of the SW differential λ_{SW} .

In what follows we will therefore concentrate on solving the problem for the SW periods within the subspace of differentials with vanishing residue, as assumed in section 2.2. In order to do so, the first step is to include the differential λ_{SW} as a basis vector by means of a change of basis. From equations 2.2 and 2.7 we have

$$\lambda_{SW} = \sum_{j=0}^n \left(\frac{k}{2} - j\right) u_j x^j \frac{dx}{y} \quad (2.23)$$

We observe that λ_{SW} is never of the third kind, because $u_g = u_{n-1} = 0$. As $k < 2n$ and $u_n = 1$, λ_{SW} always carries a nonzero component along $x^{g+1}dx/y$, so we can take the new basis of differentials of the first and second kinds to be spanned by

$$x^i \frac{dx}{y}, \quad \text{for } i \in \{0, 1, \dots, g-1\}, \quad \lambda_{SW}, \quad x^j \frac{dx}{y}, \quad \text{for } j \in \{g+2, \dots, 2g\} \quad (2.24)$$

We will find it convenient to arrange the new basic differentials in this order. Call K the matrix implementing this change of basis from the original one in equations 2.3 and 2.4 to the above in equation 2.24; one easily checks that $\det K \neq 0$. If ω and π are column vectors representing the old and new basic differentials, respectively, then from the matrix expression

$$K \cdot \omega = \pi \quad (2.25)$$

there follows a similar relation for the corresponding periods,

$$K \cdot \Omega = \Pi \quad (2.26)$$

where Π denotes the periods associated with the new basic differentials, *i.e.*, those defined in equation 2.24. Converting equation 2.22 to the new basis is immediate:

$$\frac{\partial}{\partial u_i} \Pi^{(-1/2)} = \left[K \cdot D(u_i) \cdot M^{-1} \cdot K^{-1} + \frac{\partial K}{\partial u_i} \cdot K^{-1} \right] \Pi^{(-1/2)} \quad (2.27)$$

Finally, define U_i to be

$$U_i := \left[K \cdot D(u_i) \cdot M^{-1} \cdot K^{-1} + \frac{\partial K}{\partial u_i} \cdot K^{-1} \right] \quad (2.28)$$

in order to reexpress equation 2.27 as

$$\frac{\partial}{\partial u_i} \Pi^{(-1/2)} = U_i \cdot \Pi^{(-1/2)} \quad (2.29)$$

The matrix U_i is computable from the above; its entries are rational functions of the moduli u_i .

The invertibility of M remains to be addressed. Clearly, as the definition of the M matrix requires restriction to an appropriate subspace of differentials, this issue will have to be dealt with on a case-by-case basis. However, some general arguments can be put forward. From [29] we have the following decomposition for the discriminant $\Delta(u_i)$ of the curve:

$$\Delta(u_i) = a(x)W(x) + b(x)\frac{\partial W(x)}{\partial x} \quad (2.30)$$

where $a(x)$ and $b(x)$ are certain polynomials in x . This property is used in [11] as follows. Taking the modular derivative $\partial/\partial u_i$ of the period integral causes the power in the denominator to increase by one unit, as in equation 2.20 $\mu + 1 \rightarrow \mu + 2$. In [11], this exponent is made to decrease again by use of the formula

$$\frac{\phi(x)}{W^{\mu/2}} = \frac{1}{\Delta(u_i)} \frac{1}{W^{\mu/2-1}} \left(a\phi + \frac{2}{\mu-2} \frac{d}{dx}(b\phi) \right) \quad (2.31)$$

where $\phi(x)$ is any polynomial in x . Equation 2.31 is valid only under the integral sign. It ceases to hold when the curve is singular, *i.e.*, at those points of moduli space such that $\Delta(u_i) = 0$. The defining equation of the M matrix, 2.19, is equivalent to equation 2.31, when the latter is read from right to left, *i.e.*, in decreasing order of μ . We therefore expect M to be invertible except at the singularities of moduli space, *i.e.*, on the zero locus of $\Delta(u_i)$. A proof of this fact is given in appendix A.

To further elaborate on the above argument, let us observe that the homology cycles of $H_1(\Sigma_g)$ are defined so as to encircle the zeroes of W . A vanishing discriminant $\Delta(u_i) = 0$ at some given point of moduli space implies the vanishing of the homology cycle that encircles the two collapsing roots, *i.e.*, a degeneration of Σ_g . With this vanishing cycle there is also some differential in the cohomology of Σ_g disappearing as well. We therefore expect the PF equations to exhibit some type of singular behaviour when $\Delta(u_i) = 0$, as they in fact do.

Equation 2.29 is the most general expression that one can derive without making any specific assumption as to the nature of the gauge group or the (massless) matter content of the theory. From now on, however, a case-by-case analysis is necessary, as required by the different gauge groups. This is natural, since the SW differential depends on the choice of a gauge group and matter content. However, some general features do emerge, which allow one to observe a general pattern, as will be explained in the following section.

3 Decoupling the Picard-Fuchs Equations

3.1 The B_r and D_r gauge groups

Let us first consider the $SO(2r + 1)$ and $SO(2r)$ gauge theories⁶, either for the pure SYM case, or in the presence of massless matter hypermultiplets in the fundamental representation. We restrict ourselves to asymptotically free theories. From [4, 10, 16], the polynomial $p(x)$ of equation 2.2 is even, as $u_{2j+1} = 0$. Therefore, the curves 2.1 describing moduli space are invariant under an $x \rightarrow -x$ symmetry. This invariance is a consequence of two facts: a \mathbf{Z}_2 factor present in the corresponding Weyl groups, which causes the odd Casimir operators of the group to vanish, and the property that the Dynkin index of the fundamental representation is even. This symmetry turns out to be useful in decoupling the PF equations, as it determines the right subspace of differentials that one must restrict to.

Call a differential $\omega_m = x^m dx/y$ *even* (respectively, *odd*) if m is even (respectively, odd).⁷ We will thus talk about even or odd periods accordingly. From equation 2.23 we have that λ_{SW} is even for these gauge groups. One also sees from equations 2.17 and 2.18 that the recursion relations involved in deriving the matrices $D(u_i)$ and M do not mix even with odd periods, as they always have a step of two units. This is a natural decoupling which strongly suggests omitting the odd and restricting to the even periods, something we henceforth do. In particular, the matrix M of equation 2.19 will also be restricted to this even subspace; we will check that $\det M$ then turns out to be proportional to (some powers of) the factors of $\Delta(u_i)$. The genus $g = n - 1$ is always odd, so the periods $\Omega_{n-1}^{(\pm 1/2)}$ do not appear after such a restriction. Another consequence is that the values $m = n - 1$ and $m = 3n - 1$ at which equations 2.17 and 2.18 blow up are automatically jumped over by the recursions.

The even basic differentials of equation 2.24 are

$$\frac{dx}{y}, \quad x^2 \frac{dx}{y}, \quad \dots, x^{g-1} \frac{dx}{y}, \quad \lambda_{SW}, \quad x^{g+3} \frac{dx}{y}, \dots, x^{2g} \frac{dx}{y} \quad (3.1)$$

and there are n of them. We have that n itself is even, *i.e.*, the subspace of even differentials

⁶Although there exists a well defined relation between the rank r and the genus $g = n - 1$ of the corresponding curve, we will not require it, and will continue to use n , rather than its expression as a function of r . For the gauge groups in this section, we have $g = 2r - 1$.

⁷This definition excludes the dx piece of the differential; thus, *e.g.* $x dx/y$ is defined to be odd. Obviously, this is purely a matter of convention.

is even-dimensional, so $n = 2s$ for some s .⁸ According to the notation introduced in equation 2.25, let us denote the basic differentials of equation 3.1 by

$$\{\pi_1, \dots, \pi_s, \pi_{s+1} = \lambda_{SW}, \pi_{s+2}, \dots, \pi_{2s}\} \quad (3.2)$$

where, for the sake of clarity, indices have been relabelled so as to run from 1 to $2s$. In equation 3.2, all differentials preceding $\lambda_{SW} = \pi_{s+1}$ are of the first kind; from λ_{SW} onward, all differentials are of the second kind. The periods corresponding to the differentials of equation 3.2 are

$$\{\Pi_1, \dots, \Pi_s, \Pi_{s+1} = \Pi_{SW}, \Pi_{s+2}, \dots, \Pi_{2s}\} \quad (3.3)$$

Notice that this restriction to the even subspace is compatible with the change of basis implemented by K , *i.e.*, K itself did not mix even with odd differentials.

Once restricted to the even subspace, equation 2.29 reads

$$\frac{\partial}{\partial u_i} \begin{pmatrix} \Pi_1 \\ \vdots \\ \Pi_s \\ \Pi_{s+1} \\ \vdots \\ \Pi_{2s} \end{pmatrix} = \begin{pmatrix} U_{11}^{(i)} & \dots & U_{1s}^{(i)} & U_{1s+1}^{(i)} & \dots & U_{12s}^{(i)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ U_{s1}^{(i)} & \dots & U_{ss}^{(i)} & U_{ss+1}^{(i)} & \dots & U_{s2s}^{(i)} \\ 0 & 1 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ U_{2s1}^{(i)} & \dots & U_{2ss}^{(i)} & U_{2ss+1}^{(i)} & \dots & U_{2s2s}^{(i)} \end{pmatrix} \begin{pmatrix} \Pi_1 \\ \vdots \\ \Pi_s \\ \Pi_{s+1} \\ \vdots \\ \Pi_{2s} \end{pmatrix} \quad (3.4)$$

where the $(s+1)$ -th row is everywhere zero, except at the i -th position, $1 \leq i \leq s$, whose entry is 1, so that

$$\frac{\partial}{\partial u_i} \Pi_{s+1} = \frac{\partial}{\partial u_i} \Pi_{SW} = \Pi_i, \quad 1 \leq i \leq s \quad (3.5)$$

Equation 3.5 follows from the property that the SW differential $\lambda_{SW} = \pi_{s+1}$ is a *potential* for the even holomorphic differentials, *i.e.*,

$$\frac{\partial \lambda_{SW}}{\partial u_i} = \frac{\partial \pi_{s+1}}{\partial u_i} = \pi_i, \quad 1 \leq i \leq s \quad (3.6)$$

since integration of equation 3.6 along some 1-cycle produces the corresponding statement for the periods. An analogous property for the odd periods does not hold, as all the odd moduli u_{2i+1} vanish by symmetry.

⁸The precise value of s can be given as a function of the gauge group, *i.e.*, as a function of n , but it is irrelevant to the present discussion.

To proceed further, consider the U_i matrix in equation 3.4 and block-decompose it as

$$U_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \quad (3.7)$$

where all four blocks A_i , B_i , C_i and D_i are $s \times s$. Next take the equations for the derivatives of holomorphic periods $\partial \Pi_j / \partial u_i$, $1 \leq j \leq s$, and solve them for the meromorphic periods Π_j , $s \leq j \leq 2s$, in terms of the holomorphic ones and their modular derivatives. That is, consider

$$\frac{\partial}{\partial u_i} \begin{pmatrix} \Pi_1 \\ \vdots \\ \Pi_s \end{pmatrix} - A_i \begin{pmatrix} \Pi_1 \\ \vdots \\ \Pi_s \end{pmatrix} = B_i \begin{pmatrix} \Pi_{s+1} \\ \vdots \\ \Pi_{2s} \end{pmatrix} \quad (3.8)$$

Solving equation 3.8 for the meromorphic periods involves inverting the matrix B_i . Although we lack a formal proof that B_i is invertible, $\det B_i$ turns out to vanish on the zero locus of the discriminant of the curve in all the cases catalogued in appendix B, so B_i will be invertible except at the singularities of moduli space. From equation 3.8,

$$\begin{pmatrix} \Pi_{s+1} \\ \vdots \\ \Pi_{2s} \end{pmatrix} = B_i^{-1} \cdot \left(\frac{\partial}{\partial u_i} - A_i \right) \begin{pmatrix} \Pi_1 \\ \vdots \\ \Pi_s \end{pmatrix} \quad (3.9)$$

We are interested in the SW period Π_{s+1} only, so we discard all equations in 3.9 but the first one:

$$\Pi_{s+1} = (B_i^{-1})_1^r \frac{\partial \Pi_r}{\partial u_i} - (B_i^{-1} A_i)_1^r \Pi_r \quad (3.10)$$

where a sum over r , $1 \leq r \leq s$, is implicit in equation 3.10. Finally, as the right-hand side of equation 3.10 involves nothing but holomorphic periods and modular derivatives thereof, we can use 3.5 to obtain an equation involving the SW period $\Pi_{s+1} = \Pi_{SW}$ on both sides:

$$\Pi_{SW} = (B_i^{-1})_1^r \frac{\partial^2 \Pi_{SW}}{\partial u_i \partial u_r} - (B_i^{-1} A_i)_1^r \frac{\partial \Pi_{SW}}{\partial u_r} \quad (3.11)$$

Equation 3.11 is a partial differential equation, second-order in modular derivatives, which is completely decoupled, *i.e.*, it involves nothing but the SW period Π_{SW} . The number of such equations equals the number of moduli; thus giving a decoupled system of partial, second-order differential equations satisfied by the SW period: the desired PF equations for the $N = 2$ theory.

3.2 The C_r gauge groups

Let us consider the $N_f > 0$ $Sp(2r)$ gauge theory as described by the curves given in [15, 25].^{9,10} As the Weyl group of $Sp(2r)$ contains a \mathbf{Z}_2 factor, all the odd Casimir operators vanish, *i.e.*, we have $u_{2j+1} = 0$. However, a close examination reveals that the curves given in [15, 25] contain a factor of x^2 in the left-hand side. Pulling this factor to the right-hand side has the effect of causing the resulting polynomial $p(x)$ to be *odd* under $x \rightarrow -x$. The genus $g = n - 1$ will now be even because the degree n of this resulting $p(x)$ will be odd. The \mathbf{Z}_2 symmetry dictated by the Weyl group is *not* violated, as the complete curve $y^2 = p^2(x) - x^k \Lambda^l$ continues to be even, since $k = 2(N_f - 1)$ is also even [15].

From this one can suspect that the right subspace of differentials (or periods) that one must restrict to is given by the odd differentials of equation 2.24. That this is so is further confirmed by the fact that the recursion relations 2.17 and 2.18 now have a step of 2 units, and that the SW differential λ_{SW} will now be odd, as revealed by equations 2.7 and 2.23. In consequence, the values $m = n - 1$ and $m = 3n - 1$ at which the recursions 2.17 and 2.18 blow up are jumped over, and the periods $\Omega_{n-1}^{(\pm 1/2)}$ do not appear. As was the case for the $SO(2r)$ and $SO(2r + 1)$ groups, this subspace of odd differentials is even dimensional. Furthermore, the change of basis implemented by the matrix K of equation 2.25 respects this even-odd partition, since λ_{SW} is odd.

One technical point that appears for $Sp(2r)$, but not for the orthogonal gauge groups, is the following. Let us remember that $m = 2g = 2n - 2$ is the highest value of m such that $m \in R$. We would therefore expect the period $\Omega_{2n-1}^{(+1/2)}$ to be expressible in terms of some $\Omega_m^{(+1/2)}$ with lower values of m , according to equation 2.18. However, we cannot use equation 2.18 to obtain this linear combination, since the derivation of the latter relation formally involved division by zero when one takes $m = 0$.¹¹ Instead, we must return to equation 2.11 and set $m = 0$ to arrive at

$$\Omega_{2n-1}^{(+1/2)} = -\frac{1}{2n} \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} (j+l) u_j u_l \Omega_{j+l-1}^{(+1/2)} - \frac{1}{n} \sum_{j=0}^{n-1} (n+j) u_j \Omega_{n+j-1}^{(+1/2)} + \frac{k}{2n} \Omega_{k-1}^{(+1/2)} \quad (3.12)$$

⁹The pure $Sp(2r)$ SYM theory can be described by a curve whose polynomial $p(x)$ is even [9], so it can be studied by the methods of section 3.1

¹⁰For $Sp(2r)$, we have $g = 2r$.

¹¹Remember that m was supposed to be non-zero in passing from equation 2.11 to 2.12.

As $2n - 1$ is odd, this period was omitted from the computations of the previous section, but it will be required for the resolution of the recursion relations for $Sp(2r)$.

With this proviso, the same arguments explained for the $SO(2r)$ and $SO(2r+1)$ groups in section 3.1 hold throughout, with the only difference that we will be working in the subspace of odd periods of the first and second kinds. As a consequence, the SW differential λ_{SW} will be a potential for the odd differentials only.

3.3 The A_r gauge groups

The Weyl group of $SU(r+1)$ does not possess a \mathbf{Z}_2 factor for $r > 1$.^{12,13} This implies the existence of even as well as odd Casimir operators for the group. Correspondingly, the characteristic polynomial $p(x)$ of equation 2.2 will also have non-zero odd moduli u_{2j+1} . In general, the SW differential will be neither even nor odd, as equation 2.7 reveals. The same will be true for the polynomial $p(x)$. The $x \rightarrow -x$ symmetry used in the previous sections to decouple the PF equations, be it under its even or under its odd presentation, is spoiled.

The first technical consequence of the above is that equation 3.12 will have to be taken into consideration when solving the recursion relations 2.17 and 2.18, because the period $\Omega_{2n-1}^{(+1/2)}$ will be required. Moreover, we have learned that an essential point to be addressed is the identification of the appropriate subspace of differentials (or periods) that we must restrict to. It turns out that the recursion relation 2.18 must eventually be evaluated at $m = 3n - 1$. To prove this assertion, consider the value $m = 2n - 2$ in equation 2.17, which is allowed since we still have $2g = 2n - 2 \in R$. From the right-hand side of this equation we find that, whenever $j + l = n + 1$, the period $\Omega_{3n-1}^{(+1/2)}$ is required. However, equation 2.18 blows up when $m = 3n - 1$. We have seen that this problem did not occur for the orthogonal and symplectic gauge groups.

The origin of this difficulty can be traced back to the fact that, in the sequence of differentials of the first and second kind given in equations 2.3 and 2.4, there is a gap at $m = g$, since $x^g dx/y$ is always a differential of the third kind. As the recursion relations now have a step of one unit, we cannot jump over the value $m = g = n - 1$. To clarify this point, let us

¹²Obviously, $SU(2)$ is an exception to this discussion. The corresponding PF equations are very easy to derive and to decouple for the SW period. See, *e.g.*, [20, 21, 22, 24].

¹³For $SU(r+1)$ we have $r = g$.

give an alternative expression for $\Omega_{2n-2}^{(-1/2)}$ that will bear this out.

Consider $x^{2g}W = x^{2n-2}W$ as a polynomial in x , and divide it by $\partial W/\partial x$ to obtain a certain quotient $q(x)$, plus a certain remainder $r(x)$:

$$x^{2n-2}W(x) = q(x)\frac{\partial W(x)}{\partial x} + r(x) \quad (3.13)$$

The coefficients of both $q(x)$ and $r(x)$ will be certain polynomial functions in the moduli u_i , explicitly obtainable from 3.13. The degree of $r(x)$ in x will be $2n-2$, while that of $q(x)$ will be $2n-1$, so let us put

$$q(x) = \sum_{j=0}^{2n-1} q_j(u_i)x^j, \quad r(x) = \sum_{l=0}^{2n-2} r_l(u_i)x^l \quad (3.14)$$

We have

$$W(x) = x^{2n} + \dots, \quad \frac{\partial W(x)}{\partial x} = 2nx^{2n-1} + \dots, \quad x^{2n-2}W(x) = x^{4n-2} + \dots \quad (3.15)$$

so $q(x)$ must be of the form

$$q(x) = \frac{1}{2n}x^{2n-1} + \dots \quad (3.16)$$

Furthermore, from equation 3.13,

$$\begin{aligned} - (1+\mu)\Omega_{2n-2}^{(\mu)} &= (-1)^{\mu+2}\Gamma(\mu+2) \oint_{\gamma} \frac{x^{2n-2}W}{W^{\mu+2}} = \\ &= (-1)^{\mu+2}\Gamma(\mu+2) \oint_{\gamma} \frac{1}{W^{\mu+2}} \left[\sum_{j=0}^{2n-1} q_j(u_i)x^j \frac{\partial W}{\partial x} + \sum_{l=0}^{2n-2} r_l(u_i)x^l \right] \end{aligned} \quad (3.17)$$

Integrate by parts in the first summand of equation 3.17 to obtain

$$- (1+\mu)\Omega_{2n-2}^{(\mu)} = - \sum_{j=0}^{2n-1} jq_j(u_i)\Omega_{j-1}^{(\mu)} + \sum_{l=0}^{2n-2} r_l(u_i)\Omega_l^{(\mu+1)} \quad (3.18)$$

Now set $\mu = -1/2$ in equation 3.18 and solve for $\Omega_{2n-2}^{(-1/2)}$ using 3.16:

$$\Omega_{2n-2}^{(-1/2)} = \frac{2n}{n-1} \left(\sum_{l=0}^{2n-2} r_l \Omega_l^{(+1/2)} - \sum_{j=0}^{2n-2} jq_j \Omega_{j-1}^{(-1/2)} \right) \quad (3.19)$$

Clearly, the right-hand side of equation 3.19 will in general involve the periods $\Omega_{n-1}^{(\pm 1/2)}$ corresponding to the gap in the sequence that defines the basic range R . In principle, this implies that the subspace of differentials we must restrict to is that of the ω_m with $m \in R \cup \{n-1\}$.

Let us recall that M is the matrix of coefficients in the expansion of $\Omega_m^{(-1/2)}$ as linear functions of the $\Omega_m^{(+1/2)}$. Inclusion of $\Omega_{n-1}^{(\pm 1/2)}$ would, in principle, increase the number of rows and columns of M by one unit, the increase being due to the expansion of $\Omega_{n-1}^{(-1/2)}$ as a linear combination of the $\Omega_m^{(+1/2)}$, where $m \in R \cup \{n-1\}$. However, we have no such expansion at hand. We cannot define M as a $(2g+1) \times (2g+1)$ matrix; the best we can have is $2g$ rows (corresponding to the $2g$ periods $\Omega_m^{(-1/2)}$, where $m \in R$), and $2g+1$ columns (corresponding to the $2g+1$ periods $\Omega_m^{(+1/2)}$, where $m \in R \cup \{n-1\}$). As a non-square matrix cannot be invertible, this seems to imply the need to restrict ourselves to a $2g \times 2g$ submatrix with maximal rank, and look for an invertible M matrix on that subspace. The procedure outlined in what follows serves precisely that purpose:

- Use equations 2.17 and 2.18 to express $\Omega_m^{(-1/2)}$, where $m \neq n-1$, as linear combinations of the $\Omega_m^{(+1/2)}$, where $m \in R \cup \{n-1\}$, plus possibly also of $\Omega_{n-1}^{(-1/2)}$. That the latter period can appear in the right-hand side of these expansions has already been illustrated in equation 3.19. This gives us a $2g \times (2g+1)$ matrix.

- Any occurrence of $\Omega_{n-1}^{(-1/2)}$ in the expansions that define the $2g$ rows of M is to be transferred to the left-hand side of the equations. Such occurrences will only happen when $m > n-1$ as, for $m < n-1$, equations 2.17 and 2.18 do not involve $\Omega_{n-1}^{(\pm 1/2)}$. Transferring $\Omega_{n-1}^{(-1/2)}$ to the left will also affect the $D(u_i)$ matrices of equation 2.21: whenever the left-hand side presents occurrences of $\Omega_{n-1}^{(-1/2)}$ (with u_i -dependent coefficients), the corresponding modular derivatives will have to be modified accordingly.

- As the number of columns of M will exceed that of rows by one, a linear dependence relation between the $\Omega_m^{(+1/2)}$ is needed. This will have the consequence of effectively reducing M to a *square* matrix. Only so will it have a chance of being invertible, as required by the preceding sections.

In what follows we will derive the sought-for linear dependence relation between the $\Omega_m^{(+1/2)}$, where $m \in R \cup \{n-1\}$. The procedure is completely analogous to that used in equations 3.13 to 3.19. Consider $x^{n-1}W$ as a polynomial in x , and divide it by $\partial W/\partial x$ to obtain a certain quotient $\tilde{q}(x)$, plus a certain remainder $\tilde{r}(x)$, whose degrees are n and $2n-2$, respectively:

$$\tilde{q}(x) = \sum_{j=0}^n \tilde{q}_j(u_i) x^j, \quad \tilde{r}(x) = \sum_{l=0}^{2n-2} \tilde{r}_l(u_i) x^l \quad (3.20)$$

By the same arguments as in equations 3.15 and 3.16, we can write

$$\tilde{q}(x) = \frac{1}{2n}x^n + \dots \quad (3.21)$$

Furthermore, following the same reasoning as in equations 3.17 and 3.18, we find

$$\begin{aligned} - (1 + \mu)\Omega_{n-1}^{(\mu)} &= (-1)^{\mu+2}\Gamma(\mu+2) \oint_{\gamma} \frac{x^{n-1}W}{W^{\mu+2}} = \\ &= - \sum_{j=0}^n j\tilde{q}_j(u_i)\Omega_{j-1}^{(\mu)} + \sum_{l=0}^{2n-2} \tilde{r}_l(u_i)\Omega_l^{(\mu+1)} \end{aligned} \quad (3.22)$$

Setting $\mu = -1/2$ and solving equation 3.22 for $\Omega_{n-1}^{(-1/2)}$ produces

$$0 = \left(n\tilde{q}_n - \frac{1}{2}\right)\Omega_{n-1}^{(-1/2)} = - \sum_{j=0}^{n-1} j\tilde{q}_j(u_i)\Omega_{j-1}^{(-1/2)} + \sum_{l=0}^{2n-2} \tilde{r}_l(u_i)\Omega_l^{(+1/2)} \quad (3.23)$$

where equation 3.21 has been used to equate the left-hand side to zero. We therefore have

$$\sum_{j=0}^{n-1} j\tilde{q}_j(u_i)\Omega_{j-1}^{(-1/2)} = \sum_{l=0}^{2n-2} \tilde{r}_l(u_i)\Omega_l^{(+1/2)} \quad (3.24)$$

Equation 3.24 is a linear dependence relation between $\Omega^{(-1/2)}$ and $\Omega^{(+1/2)}$ which does not involve $\Omega_{n-1}^{(-1/2)}$. Therefore, we are now able to make use of equations 2.17 and 2.18, *i.e.*, of the allowed rows of M , to recast the left-hand side of equation 3.24 as a linear combination of the $\Omega^{(+1/2)}$:

$$\sum_{j=0}^{n-1} j\tilde{q}_j(u_i) \sum_{r \in R} [M]_{j-1}^r \Omega_r^{(+1/2)} = \sum_{l=0}^{2n-2} \tilde{r}_l(u_i)\Omega_l^{(+1/2)} \quad (3.25)$$

Equation 3.25 is the sought-for linear dependence relation that appears due to the inclusion of $\Omega_{n-1}^{(\pm 1/2)}$. Restriction to the subspace defined by this relation produces a *square*, $2g \times 2g$ matrix:

$$\tilde{\Omega}^{(-1/2)} = \tilde{M}\tilde{\Omega}^{(+1/2)} \quad (3.26)$$

The tildes in the notation remind us that the left-hand side will include occurrences of $\Omega_{n-1}^{(-1/2)}$ when $m > n - 1$, possibly multiplied by some u_i -dependent coefficients, while the right-hand side has been reduced as dictated by the linear dependence relation 3.25. *We claim that the \tilde{M} matrix so defined is invertible, its determinant vanishing on the zero locus of the discriminant $\Delta(u_i)$ of the curve.* It is on this $2g$ -dimensional space of differentials (or periods) that we will be working.

Let us make some technical observations on the procedure just described. In practice, restriction to the subspace determined by equation 3.25 means solving for some given $\Omega_m^{(+1/2)}$, where $m \in R \cup \{n-1\}$, as a linear combination (with u_i -dependent coefficients) of the rest. The particular $\Omega_m^{(+1/2)}$ that can be solved for depends on the coefficients entering equation 3.25; any period whose coefficient is non-zero will do. Obviously, the particular $\Omega_m^{(+1/2)}$ that is being solved for in equation 3.25 is irrelevant (as long as its coefficient is non-zero), since any such $\Omega_m^{(+1/2)}$ so obtained is just a different, but equivalent, expression of the same linear relation 3.25. Whatever the choice, $\det \tilde{M}$ will continue to vanish on the zero locus of $\Delta(u_i)$.

However, differences may arise in the actual entries of \tilde{M} , due to the fact that different (though equivalent) sets of basic $\Omega_m^{(+1/2)}$ are being used. Once a given set of $2g$ independent $\Omega_m^{(+1/2)}$ has been picked, *i.e.*, after imposing equation 3.25, this one set must be used throughout. In particular, the right-hand sides of equations 2.21 will also have to be expressed in this basis. As the $\Omega_m^{(+1/2)}$ disappear from the computations already at the level of equation 2.22, the particular choice made is irrelevant. For the same reason, one can easily convince oneself that the final PF equations obtained are independent of the actual choice made.

Let us point out two further consequences of this prescription used to define \tilde{M} . First, some of the equations collected in 2.22 may possibly involve, both in the right and the left-hand sides, occurrences of $\Omega_{n-1}^{(-1/2)}$ and its modular derivatives, as dictated by the prescription. One might worry that the latter will not disappear from the final result for the SW differential λ_{SW} . That it will always drop out follows from the fact that none of the first g equations in 2.22 involves $\Omega_{n-1}^{(-1/2)}$, as they are untouched by the defining prescription of \tilde{M} . The decoupling procedure followed to decouple λ_{SW} also respects this property, as it basically discards all equations for the periods of non-holomorphic differentials (with the exception of the SW differential itself, of course).

A second consequence of the prescription used to define \tilde{M} is the fact that its entries may now become *rational* functions of the moduli, rather than *polynomial* functions. This is different from the situation for the $SO(2r+1)$, $Sp(2r)$ and $SO(2r)$ gauge groups, where these entries were always polynomials in the u_i . The reason is that solving the linear relation 3.25 for one particular $\Omega_m^{(+1/2)}$ may involve division by a polynomial in the u_i .

Having taken care of the difficulty just mentioned, *i.e.*, the identification of the appropriate space of periods on which \tilde{M} will be invertible, the rest of the decoupling procedure

already explained for the $SO(2r+1)$, $Sp(2r)$ and $SO(2r)$ gauge groups holds throughout. In particular, expressions totally analogous to those from equation 3.4 to 3.11 continue to be valid, with $s = g$. As a compensation for this technical difficulty of having nonzero odd Casimir operators, one has that the SW differential truly becomes a potential for *all* holomorphic differentials on the curve, so the equivalent of equation 3.5 now includes the odd holomorphic periods as well.

3.4 The exceptional gauge groups

The method developed in section 2 can also be applied to obtain the PF equations associated with $N = 2$ SYM theories (with or without massless matter) when the gauge group is an exceptional group, as the vacuum structure of these theories is also described by hyperelliptic curves [8, 9, 16]. In principle, a set of (1st-order) PF equations similar to those given in equation 2.29 can also be derived. However, we have seen that the decoupling procedure described in section 3 depends crucially on our ability to identify an appropriate subspace of periods to restrict to. Such an identification makes use of the structure of the corresponding Weyl group. With the exception of G_2 , whose Weyl group is D_6 (the dihedral group of order 12), the Weyl groups of F_4 , E_6 , E_7 and E_8 are not easily manageable, given their high orders. Therefore, we cannot hope to be able to develop a systematic decoupling prescription similar to the one given for the classical gauge groups. This is just a reflection of the exceptionality of the groups involved.

Another difficulty, which we have illustrated in the particular case of G_2 below, is the fact that the genus g of the corresponding Riemann surface Σ_g will in general be too high compared with the number of independent Casimir operators. As there is one modulus u_i per Casimir operator, we cannot expect the SW differential to be a potential for *all* g holomorphic differentials on Σ_g . This fact has already been observed for the B_r , C_r and D_r gauge groups. However, the novelty here is that, in general, the best one can hope for is to equate $\partial\lambda_{SW}/\partial u_i$ to some linear combination (with u_i -dependent coefficients) of a number of holomorphic differentials ω_j ,

$$\frac{\partial\lambda_{SW}}{\partial u_i} = \sum_{j=0}^{g-1} c_i^j(u_l)\omega_j \quad (3.27)$$

Although the requirement of homogeneity with respect to the (residual) R -symmetry can give us a clue as to the possible terms that can enter the right-hand side of equation 3.27, the actual linear combinations can only be obtained by explicit computation. In general, such linear combinations may involve more than one non-zero coefficient $c_i^j(u_l)$. As a consequence, the decoupling procedure explained in previous sections breaks down, since it hinged on the SW differential λ_{SW} being a potential for (some well defined subspace of) the holomorphic differentials, *i.e.*, on all $c_i^j(u_l)$ but one being zero. In other words, even if it were possible to identify the appropriate subspace of periods that one must restrict to, the high value of the genus g would probably prevent a decoupling of the PF equations.

As an illustration, we have included the details pertaining to G_2 in section 4.3.

4 Examples

4.1 Pure $SO(5)$ SYM theory

The vacuum structure of the effective, pure $N = 2$ SYM theory with gauge group $SO(5)$ is described by the curve [10]

$$W = y^2 = p(x)^2 - x^2 = x^8 + 2ux^6 + (u^2 + 2t)x^4 + 2tux^2 + t^2 - x^2 \quad (4.1)$$

where $p(x) = x^4 + ux^2 + t$. The quantum scale has been set to unity, $\Lambda = 1$, and the moduli u and t can be identified with the second- and fourth-order Casimir operators of $SO(5)$, respectively. The discriminant $\Delta(u, t)$ is given by

$$\Delta(u, t) = 256t^2(-27 + 256t^3 + 144tu - 128t^2u^2 - 4u^3 + 16tu^4)^2 \quad (4.2)$$

Equation 4.1 describes a hyperelliptic Riemann surface of genus $g = 3$, Σ_3 . The holomorphic differentials on Σ_3 are dx/y , xdx/y and x^2dx/y , while x^4dx/y , x^5dx/y and x^6dx/y are meromorphic differentials of the second kind. From equation 2.7, the SW differential is given by

$$\lambda_{SW} = (-3x^4 - ux^2 + t)\frac{dx}{y} \quad (4.3)$$

Both $p(x)$ and λ_{SW} are even under $x \rightarrow -x$.¹⁴ We therefore restrict ourselves to the subspace of differentials on Σ_3 spanned by $\{dx/y, x^2dx/y, x^4dx/y, x^6dx/y\}$. This is further confirmed

¹⁴Recall that our convention leaves out the dx term in the differential.

by the fact that the recursion relations 2.17 and 2.18 now have a step of 2 units,

$$\Omega_n^{(-1/2)} = \frac{8}{n-3} \left[t^2 \Omega_n^{(+1/2)} + \frac{3}{4} (2tu - 1) \Omega_{n+2}^{(+1/2)} + \frac{1}{2} (u^2 + 2t) \Omega_{n+4}^{(+1/2)} + \frac{u}{2} \Omega_{n+6}^{(+1/2)} \right] \quad (4.4)$$

and

$$\begin{aligned} \Omega_n^{(+1/2)} &= \frac{1}{n-11} \left[(10-n) 2u \Omega_{n-2}^{(+1/2)} + (9-n) (u^2 + 2t) \Omega_{n-4}^{(+1/2)} \right. \\ &\quad \left. + (8-n) (2tu - 1) \Omega_{n-6}^{(+1/2)} + (7-n) t^2 \Omega_{n-8}^{(+1/2)} \right] \end{aligned} \quad (4.5)$$

so that even and odd values don't mix. The solution of these recursions can be given in terms of the initial data $\{\Omega_0^{(+1/2)}, \Omega_2^{(+1/2)}, \Omega_4^{(+1/2)}, \Omega_6^{(+1/2)}\}$, where the indices take on the values allowed by the even subspace of differentials. From equations 4.4 and 4.5, the M matrix of equation 2.19 can be readily computed. Its determinant is found to be a product of powers of the factors of the discriminant $\Delta(u, t)$:

$$\det M = \frac{16}{9} t^2 (-27 + 256t^3 + 144tu - 128t^2u^2 - 4u^3 + 16tu^4) \quad (4.6)$$

Therefore, it has the same zeroes as $\Delta(u, t)$ itself, but with different multiplicities.

Next, the change of basis in the space of differentials required by equation 2.25 is effected by the matrix

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t & -u & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.7)$$

The matrices $D(u)$ and $D(t)$ defined in equations 2.20 and 2.21 can be computed using the expressions for W and the recursion relations given in equations 4.4 and 4.5.¹⁵ Once $D(u)$ and $D(t)$ are reexpressed in the new basis $\{\pi_0, \pi_2, \pi_4 = \lambda_{SW}, \pi_6\}$ defined by K in 4.7, they produce the U_i matrices of equation 2.28. Let us just observe that, from the corresponding third rows of U_t and U_u , one finds

$$\frac{\partial \Pi_4}{\partial t} = \Pi_0, \quad \frac{\partial \Pi_4}{\partial u} = \Pi_2 \quad (4.8)$$

as expected for the SW period $\Pi_4 = \Pi_{SW}$.

¹⁵For the sake of brevity, the explicit expressions of these matrices are not reproduced here.

Let us consider the t -modulus and carry out the decoupling procedure for the SW period explicitly. Block-divide the U_t matrix as required by equation 3.7:

$$U_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix} \quad (4.9)$$

Specifically, one finds

$$A_t = \frac{1}{\det M} \frac{16t}{27} \begin{pmatrix} t(-384t^2 + 27u - 80tu^2 + 4u^4) & (135t - 240t^2u - 27u^2 + 68tu^3 - 4u^5) \\ 4t^2(-18 + 76tu + u^3) & 4t(60t^2 - 7tu^2 - u^4) \end{pmatrix} \quad (4.10)$$

$$B_t = \frac{1}{\det M} \begin{pmatrix} -\frac{64}{27}t(12t^2 + 27u - 47tu^2 + 4u^4) & \frac{16}{3}t(27 - 48tu + 4u^3) \\ -\frac{16}{27}t^2(9 + 160tu + 16u^3) & \frac{64}{3}t^2(12t + u^2) \end{pmatrix} \quad (4.11)$$

We observe that

$$\det B_t = \frac{256}{9}t^3(27 - 256t^3 - 144tu + 128t^2u^2 + 4u^3 - 16tu^4) \quad (4.12)$$

so B_t is invertible except at the singularities of moduli space. Carry out the matrix multiplications of equation 3.10 to get

$$\Pi_4 = -(16t^2 + \frac{4}{3}tu^2)\frac{\partial\Pi_0}{\partial t} + (9 - 16tu + \frac{4}{3}u^3)\frac{\partial\Pi_2}{\partial t} - 8t\Pi_0 \quad (4.13)$$

Finally, use 4.8 to obtain a decoupled equation for the SW period $\Pi_4 = \Pi_{SW}$:

$$\mathcal{L}_1\Pi_{SW} = 0, \quad \mathcal{L}_1 = 4t(u^2 + 12t)\frac{\partial^2}{\partial t^2} - (27 - 48tu + 4u^3)\frac{\partial^2}{\partial t\partial u} + 24t\frac{\partial}{\partial t} + 3 \quad (4.14)$$

Analogous steps for the u modulus lead to

$$\mathcal{L}_2\Pi_{SW} = 0, \quad \mathcal{L}_2 = (9 - 32tu)\frac{\partial^2}{\partial t\partial u} - 4(12t + u^2)\frac{\partial^2}{\partial u^2} - 8t\frac{\partial}{\partial t} - 1 \quad (4.15)$$

4.2 Pure $SU(3)$ SYM theory.

The vacuum structure of the effective, pure $N = 2$ SYM theory with gauge group $SU(3)$ is described by the curve [2, 3, 11]¹⁶

$$W = y^2 = p(x)^2 - 1 = x^6 + 2ux^4 + 2tx^3 + u^2x^2 + 2tux + t^2 - 1 \quad (4.16)$$

¹⁶Our moduli (u, t) correspond to $(-u, -v)$ in [2].

where $p(x) = x^3 + ux + t$. The quantum scale has been set to unity, $\Lambda = 1$, and the moduli u and t can be identified with the second- and third-order Casimir operators of $SU(3)$, respectively. The discriminant is given by

$$\Delta(u, t) = -64(27 - 54t + 27t^2 + 4u^3)(27 + 54t + 27t^2 + 4u^3) \quad (4.17)$$

Equation 4.16 describes a Riemann surface of genus $g = 2$, Σ_2 . The holomorphic differentials on Σ_2 are dx/y and $x dx/y$, while $x^3 dx/y$ and $x^4 dx/y$ are meromorphic differentials of the second kind. From equation 2.7, the SW differential is given by

$$\lambda_{SW} = -(ux + 3x^3) \frac{dx}{y} \quad (4.18)$$

The recursion relations 2.17 and 2.18 can be easily computed. They have a step of 1 unit,

$$\begin{aligned} \Omega_n^{(-1/2)} = \\ \frac{1}{n-2} \left[4u\Omega_{n+4}^{(+1/2)} + 6t\Omega_{n+3}^{(+1/2)} + 4u^2\Omega_{n+2}^{(+1/2)} + 10tu\Omega_{n+1}^{(+1/2)} + 6(t^2 - 1)\Omega_n^{(+1/2)} \right] \end{aligned} \quad (4.19)$$

and

$$\begin{aligned} \Omega_n^{(+1/2)} = \frac{1}{n-8} \left[2u(7-n)\Omega_{n-2}^{(+1/2)} + 2t\left(\frac{13}{2} - n\right)\Omega_{n-3}^{(+1/2)} \right. \\ \left. + (6-n)u^2\Omega_{n-4}^{(+1/2)} + \left(\frac{11}{2} - n\right)2tu\Omega_{n-5}^{(+1/2)} + (n-5)(1-t^2)\Omega_{n-6}^{(+1/2)} \right] \end{aligned} \quad (4.20)$$

which means that the solution for $\{\Omega_0^{(-1/2)}, \Omega_1^{(-1/2)}, \Omega_3^{(-1/2)}, \Omega_4^{(-1/2)}\}$ in terms of the initial values $\{\Omega_0^{(+1/2)}, \Omega_1^{(+1/2)}, \Omega_3^{(+1/2)}, \Omega_4^{(+1/2)}\}$ will involve both even and odd values of n . Inspection of the recursion relations above shows that, in the computation of $\Omega_4^{(-1/2)}$, the value of $\Omega_8^{(+1/2)}$ is needed. This illustrates the situation described in section 3.3.

As already explained, in order to properly identify the right subspace of periods to work with, it suffices to include $\Omega_2^{(-1/2)}$ and its counterpart $\Omega_2^{(+1/2)}$. It is then possible to use equations 4.19 and 4.20 in order to write a set of 4 equations expressing $\Omega_n^{(-1/2)}$, where $n = 0, 1, 3, 4$, in terms of $\Omega_n^{(+1/2)}$ with $n = 0, 1, 2, 3, 4$ and $\Omega_2^{(-1/2)}$. As prescribed in section 3.3, we pull all occurrences of $\Omega_2^{(-1/2)}$ to the left-hand side. In the case at hand, such occurrences take place under the form

$$\Omega_4^{(-1/2)} = -u\Omega_2^{(-1/2)} + \text{terms in } \Omega_m^{(+1/2)} \quad (4.21)$$

while the expansions for the $\Omega_m^{(-1/2)}$, $m \neq 4$, do not involve $\Omega_2^{(-1/2)}$. We therefore define \tilde{M} as the matrix of coefficients in the expansion of

$$\Omega_0^{(-1/2)}, \quad \Omega_1^{(-1/2)}, \quad \Omega_3^{(-1/2)}, \quad \Omega_4^{(-1/2)} + u\Omega_2^{(-1/2)} \quad (4.22)$$

in terms of $\Omega_n^{(+1/2)}$ with $n = 0, 1, 2, 3, 4$. Next we use the linear dependence relation between the latter periods, as derived in equation 3.25, which for the particular case at hand reads

$$\Omega_2^{(+1/2)} = -\frac{u}{3}\Omega_0^{(+1/2)} \quad (4.23)$$

We use equation 4.23 to express $\Omega_2^{(+1/2)}$ in terms of $\Omega_0^{(+1/2)}$. This choice has the computational advantage that, as the coefficient of $\Omega_2^{(+1/2)}$ in equation 4.23 is a constant, no division by a polynomial is involved, and the entries of \tilde{M} continue to be polynomial functions in u and t . From the above one can immediately conclude that the periods of equation 4.22 can be expressed as certain linear combinations, with u and t -dependent coefficients, of $\{\Omega_0^{(+1/2)}, \Omega_1^{(+1/2)}, \Omega_3^{(+1/2)}, \Omega_4^{(+1/2)}\}$. After this change of basis in the space of periods (or differentials) has been performed, the rest of the construction already explained goes through. We first confirm that \tilde{M} so defined is invertible except at the singularities of moduli space, as

$$\det \tilde{M} = \frac{4}{9}(27 - 54t + 27t^2 + 4u^3)(27 + 54t + 27t^2 + 4u^3) \quad (4.24)$$

The SW differential is included in the computations upon performing the change of basis from $\{\omega_0, \omega_1, \omega_3, \omega_4 + u\omega_2\}$ to $\{\pi_1, \pi_2, \pi_3 = \lambda_{SW}, \pi_4\}$, as given by the matrix

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -u & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.25)$$

where basis indices have also been relabelled for simplicity. The matrices $D(u)$ and $D(t)$ defined in equations 2.20 and 2.21 can be equally computed using the expressions for W and the recursion relations given above, and recast in the new basis $\{\pi_1, \pi_2, \pi_3 = \lambda_{SW}, \pi_4\}$ defined by K . This produces the U_i matrices of equation 2.28. Let us just observe that, from the third rows of U_u and U_t , one finds

$$\frac{\partial \Pi_3}{\partial t} = \Pi_1, \quad \frac{\partial \Pi_3}{\partial u} = \Pi_2 \quad (4.26)$$

as expected for the SW period $\Pi_3 = \Pi_{SW}$. Further carrying out the decoupling procedure as already prescribed yields

$$\mathcal{L}_i \Pi_{SW} = 0, \quad i = 1, 2 \quad (4.27)$$

where

$$\begin{aligned}
\mathcal{L}_1 &= (27 + 4u^3 - 27t^2) \frac{\partial^2}{\partial u^2} + 12u^2t \frac{\partial^2}{\partial u \partial t} + 3tu \frac{\partial}{\partial t} + u \\
\mathcal{L}_2 &= (27 + 4u^3 - 27t^2) \frac{\partial^2}{\partial t^2} - 36ut \frac{\partial^2}{\partial u \partial t} - 9t \frac{\partial}{\partial t} - 3
\end{aligned} \tag{4.28}$$

in complete agreement with [11], once the differences in notation have been taken into account.

4.3 Pure G_2 SYM theory.

The vacuum structure of the effective, pure $N = 2$ SYM theory with gauge group G_2 is described by the curve [8, 9, 16]

$$W = y^2 = p(x)^2 - x^4 = x^{12} + 2ux^{10} + \frac{3u^2}{2}x^8 + (2t + \frac{u^3}{2})x^6 + (\frac{u^4}{16} + 2tu - 1)x^4 + \frac{tu^2}{2}x^2 + t^2 \tag{4.29}$$

where $p(x) = x^6 + ux^4 + u^2x^2/4 + t$.¹⁷ The quantum scale has been set to unity, *i.e.*, $\Lambda = 1$, and the moduli u and t can be identified with the second- and sixth-order Casimir operators of G_2 , respectively. We observe the absence of a fourth-order Casimir operator; instead, its role is fulfilled by the *square* of the second-order one. On the curve, this is reflected in the coefficient of x^2 . This is a consequence of the exceptionality of the Lie algebra G_2 . The discriminant $\Delta(u, t)$ is given by

$$\begin{aligned}
\Delta(u, t) &= 65536t^6(-16 + 108t^2 + 72tu + 8u^2 - 2tu^3 - u^4)^2 \\
&\quad (16 + 108t^2 - 72tu + 8u^2 - 2tu^3 + u^4)^2
\end{aligned} \tag{4.30}$$

Equation 4.29 describes a hyperelliptic Riemann surface of genus $g = 5$, Σ_5 . The holomorphic differentials on Σ_5 are $x^j dx/y$, where $j \in \{0, 1, 2, 3, 4\}$, while $x^{6+j} dx/y$, with j in the same range, are meromorphic differentials of the second kind. From equation 2.7, the SW differential is given by

$$\lambda_{SW} = (2t - 2ux^4 - 4x^6) \frac{dx}{y} \tag{4.31}$$

Both $p(x)$ and λ_{SW} are even under $x \rightarrow -x$. We therefore restrict ourselves to the subspace of differentials on Σ_5 spanned by $\{dx/y, x^2 dx/y, x^4 dx/y, x^6 dx/y, x^8 dx/y, x^{10} dx/y\}$. This

¹⁷Some doubts about this curve have recently been expressed in [30]. The difficulty encountered with equation (4.36) might perhaps be circumvented with a different curve.

is further confirmed by the fact that the recursion relations 2.17 and 2.18 now have a step of 2 units,

$$\begin{aligned}\Omega_n^{(-1/2)} &= \frac{1}{n-5} \left[12t^2\Omega_n^{(+1/2)} + 5tu^2\Omega_{n+2}^{(+1/2)} + (16tu + \frac{1}{2}u^4 - 8)\Omega_{n+4}^{(+1/2)} \right. \\ &\quad \left. + (12t + 3u^3)\Omega_{n+6}^{(+1/2)} + 6u^2\Omega_{n+8}^{(+1/2)} + 4u\Omega_{n+10}^{(+1/2)} \right]\end{aligned}\quad (4.32)$$

and

$$\begin{aligned}\Omega_n^{(+1/2)} &= \frac{1}{n-17} \left[(11-n)t^2\Omega_{n-12}^{(+1/2)} + \frac{1}{2}(12-n)tu^2\Omega_{n-10}^{(+1/2)} \right. \\ &\quad + (13-n)\left(\frac{u^4}{16} + 2tu - 1\right)\Omega_{n-8}^{(+1/2)} + (14-n)\left(2t + \frac{u^3}{2}\right)\Omega_{n-6}^{(+1/2)} \\ &\quad \left. + (15-n)\frac{3}{2}u^2\Omega_{n-4}^{(+1/2)} + (16-n)2u\Omega_{n-2}^{(+1/2)} \right]\end{aligned}\quad (4.33)$$

so that even and odd values don't mix. The solution of these recursions can be given in terms of the initial data $\{\Omega_0^{(+1/2)}, \Omega_2^{(+1/2)}, \Omega_4^{(+1/2)}, \Omega_6^{(+1/2)}, \Omega_8^{(+1/2)}, \Omega_{10}^{(+1/2)}\}$, where the indices take on the values allowed by the even subspace of differentials picked above. From equations 4.32 and 4.33, the M matrix of equation 2.19 can be readily computed. Its determinant is found to be a product of powers of the factors of the discriminant $\Delta(u, t)$:

$$\det M = \frac{256}{225}t^4(-16+108t^2+72tu+8u^2-2tu^3-u^4)(16+108t^2-72tu+8u^2-2tu^3+u^4) \quad (4.34)$$

Therefore, it has the same zeroes as $\Delta(u, t)$ itself, but with different multiplicities.

Next, the change of basis required by equation 2.25 is effected by the matrix

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 2t & 0 & -2u & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.35)$$

The matrices $D(u)$ and $D(t)$ defined in equations 2.20 and 2.21 can be equally computed using the expressions for W and the recursion relations given in 4.29, 4.32 and 4.33.¹⁸ Once $D(u)$ and $D(t)$ are reexpressed in the new basis $\{\pi_0, \pi_2, \pi_4, \pi_6 = \lambda_{SW}, \pi_8, \pi_{10}\}$ defined by K

¹⁸The complete system of coupled, first-order equations is not reproduced here for the sake of brevity.

in equation 4.35, they produce the U_i matrices of equation 2.28. Let us just observe that, from the corresponding fourth rows of U_t and U_u , one finds

$$\frac{\partial \Pi_6}{\partial t} = \Pi_0, \quad \frac{\partial \Pi_6}{\partial u} = \Pi_4 + \frac{u}{2} \Pi_2 \quad (4.36)$$

Equation 4.36 is the expression of 3.27 when the gauge group is G_2 . We observe the presence of a linear combination of even holomorphic periods in the right-hand side, rather than a clear-cut correspondence between holomorphic periods (or differentials) and moduli. The presence of an additional term $u\Pi_2/2$ is a consequence of the exceptionality of G_2 . To understand this fact in more detail, we observe that G_2 has rank 2, so the number of independent moduli is therefore 2. Like any other algebra, it has a quadratic Casimir operator (associated with the u modulus). In contrast to $SO(5)$, which is also rank 2, G_2 possesses no fourth-order Casimir operator; instead, the next Casimir is of order 6. It is associated with the t modulus. There is no fourth-order Casimir operator other than the trivial one, namely, the one obtained upon squaring the quadratic one. The existence of third- and fifth-order Casimir operators is forbidden by the \mathbf{Z}_2 symmetry of the Weyl group. This leaves us with just 2 independent moduli, u and t , to enter the definition of the curve as coefficients of $p(x)$. As the latter is even and of degree 6 (as dictated by the order of the highest Casimir operator), we are clearly missing one modulus (associated with a would-be quartic Casimir), whose role is then fulfilled by the square of the quadratic one. The consequence is a smaller number of moduli (2) than would be required (3) for the SW differential λ_{SW} to serve as a potential for the even holomorphic differentials. This causes the presence of the linear combination in the right-hand side of equation 4.36, and the decoupling procedure breaks down.

5 Summary and Outlook

In this paper an alternative derivation of the Picard-Fuchs equations has been presented which is systematic and well suited for symbolic computer computations. It holds for any classical gauge group, and aims explicitly at effective $N = 2$ supersymmetric gauge theories in 4 dimensions. However, the techniques presented here may well find applications beyond these specific areas. Our method makes use of the underlying group theory in order to obtain a decoupled set of partial, second-order equations satisfied by the period integrals

of the Seiberg-Witten differential. This computational simplicity allows one to derive the PF equations for large values of the rank of the gauge group with comparatively very little effort. The inclusion of massless matter hypermultiplets is also straightforward. One of the strengths of the presentation of this paper is that the techniques studied here lend themselves to a wide variety of applications, and are not limited to the SW problem only.

More interesting than the derivation of the PF equations themselves is of course the extraction of physical information from their solutions. This topic has already been studied in the literature in a number of cases [11, 20, 21, 22, 25, 26], and provides an interesting application of our techniques, which we are currently addressing [31]. Another important extension of our work is the consideration of massive matter hypermultiplets; this poses some technical challenges which are also under investigation [31]. We hope to be able to report on these issues in the near future.

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Appendix A

Below is a proof of the statement that all zeroes of $\det M$ are also zeroes of the discriminant $\Delta(u_i)$, possibly with different multiplicities. For the sake of simplicity, we give the details pertaining to the gauge groups $SO(2r+1)$, $SO(2r)$ and $Sp(2r)$ (with or without massless matter in the fundamental representation). The proof for the $SU(r+1)$ gauge groups is slightly more involved, but should be technically analogous to the one presented below.

To this purpose we recall equations 2.30 and 2.31:

$$\Delta(u_i) = a(x)W(x) + b(x)\frac{\partial W(x)}{\partial x} \quad (\text{A.1})$$

$$\frac{\phi(x)}{W^{\mu/2}} = \frac{1}{\Delta(u_i)} \frac{1}{W^{\mu/2-1}} \left(a\phi + \frac{2}{\mu-2} \frac{d}{dx}(b\phi) \right) \quad (\text{A.2})$$

As explained in the body of the paper, equation A.2 is an equivalent expression for the inversion of M , once it has been integrated along some closed 1-cycle $\gamma \in H_1(\Sigma_g)$. Let the polynomials $a(x)$ and $b(x)$ in equation A.1 have the expansions

$$a(x) = \sum_{i=0}^s a_i x^i, \quad b(x) = \sum_{j=0}^{s'} b_j x^j \quad (\text{A.3})$$

The respective degrees s and s' of $a(x)$ and $b(x)$ are easily found to be related by $s' = s + 1$. This follows from the fact that the left-hand side of equation A.1 has degree zero in x . Moreover, $W(x)$ has degree $2n = 2g + 2$ in x . Using equation A.1, and imposing the condition that the number of unknown coefficients a_i and b_j equal the number of equations they must satisfy, one easily finds $s = 2g$ and $s' = 2g + 1$. It should be borne in mind that the coefficients a_i and b_j themselves will be polynomial functions in the moduli u_i . For the gauge groups $SO(2r+1)$, $SO(2r)$ and $Sp(2r)$ (with or without massless matter), we observe that $a(x)$ will be even as a polynomial in x , while $b(x)$ will be odd.

Let us take the polynomial $\phi(x)$ in equation A.2 to be $\phi(x) = x^m$, and set $\mu = 3$ in equation A.2 to obtain the periods $\Omega^{(\pm 1/2)}$ as used in the body of the text. Now assume the polynomial $a(x)x^m + 2(b(x)x^m)'$ has the following expansion in powers of x ,

$$a(x)x^m + 2\frac{d}{dx}(b(x)x^m) = \sum_{r=0}^{m+2g} c_r(u_i)x^r \quad (\text{A.4})$$

where the $c_r(u_i)$ are some u_i -dependent coefficients. For the $SO(2r)$ and $SO(2r+1)$ gauge groups, m can be assumed to be even, while it can be assumed odd for $Sp(2r)$. Integration

of A.2 along a closed 1-cycle $\gamma \in H_1(\Sigma_g)$ produces

$$\Omega_m^{(+1/2)} = -\frac{3}{\Delta(u_i)} \sum_{r=0}^{m+2g} c_r(u_i) \Omega_r^{(-1/2)} \quad (\text{A.5})$$

As explained, equation A.5 can be taken to define the inverse M matrix, M^{-1} . In fact, it just misses the correct definition by a minor technical point. As we let the integer m run over (the even or the odd values of) the basic range R , the subscripts on the right-hand side of equation A.5 will eventually take values outside R . We can correct this by making use of the recursion relation 2.16, in order to bring m back into R . One just has to apply equation 2.16 for $\mu = -3/2$ and substitute the value of k appropriate to the gauge group and matter content under consideration.¹⁹ The resulting linear combination of periods $\Omega_q^{(-1/2)}$ in the right-hand side has lower values of the subindex q . One easily checks that, as the degree of $a(x)x^m + 2(b(x)x^m)'$ is $2g + m$, the values attained by q in the right-hand side of equation 2.16 never become negative when m runs over (the even or the odd subspace of) R . Eventually, a linear combination (with u_i -dependent coefficients) will be obtained such that all lower indices q of the periods $\Omega_q^{(-1/2)}$ will lie within (the even or the odd subspace of) R . Now this correctly defines an inverse M matrix.

We have thus expressed the (j, l) element of M^{-1} as

$$\left[M^{-1}\right]_{jl} = \frac{1}{\Delta(u_i)} P_{jl}(u_i) \quad (\text{A.6})$$

where the $P_{jl}(u_i)$ are certain polynomial functions of the moduli u_i . On the other hand, from the definition of matrix inversion,

$$\left[M^{-1}\right]_{jl} = \frac{1}{\det M} C_{jl}(u_i) \quad (\text{A.7})$$

where C_{jl} is the matrix of cofactors of M . Obviously, not all the $C_{jl}(u_i)$ are divisible by $\det M$, as otherwise M would be invertible even when $\det M = 0$. From the equality

$$\frac{1}{\det M} C_{jl}(u_i) = \frac{1}{\Delta(u_i)} P_{jl}(u_i) \quad (\text{A.8})$$

it follows that the right-hand side of equation A.8 will have to blow up whenever the left-hand side does, *i.e.*, *all zeroes of $\det M$ are also zeroes of $\Delta(u_i)$* , possibly with different

¹⁹For N_f massless multiplets, one has $k = 2 + 2N_f$ for $SO(2r + 1)$, $k = 4 + 2N_f$ for $SO(2r)$, and $k = 2(N_f - 1)$ for $Sp(2r)$.

multiplicities. The converse need not hold, as in principle nothing prevents all the $P_{jl}(u_i)$ from simultaneously having $\Delta(u_i)$ as a common factor.

Let us finally make an observation on the above proof for the $SU(r+1)$ gauge groups. From section 3.3 in the paper, we know that there are several different, though equivalent, ways of defining the \tilde{M} matrix. The particular choice made in solving the linear dependence relation 3.25 may imply division by a non-constant polynomial $f(u_i)$ in the moduli u_i , if the period that is being solved for in equation 3.25 is multiplied by a non-constant coefficient. This has the effect of causing a power of $f(u_i)$ to appear in the determinant $\det \tilde{M}$, besides the required powers of the factors of the discriminant $\Delta(u_i)$. Obviously, only the zeroes of the latter are relevant, as they are the ones associated with the singularities of the curve. The zeros of $\det \tilde{M}$ due to the presence of $f(u_i)$ are a consequence of the prescription used to define \tilde{M} .

Appendix B

Below are listed the PF equations satisfied by the period integrals of the SW differential of a number of effective $N = 2$ SYM theories (with and without massless matter), as classified by their gauge groups. Rather than an exhaustive list, we give a sample of cases in increasing order of the rank r of the gauge group, with some examples including massless matter hypermultiplets in the fundamental representation. The hyperelliptic curves describing their corresponding vacua are also quoted for notational completeness. For computational simplicity, we systematically set the quantum scale Λ of the theory to unity, *i.e.*, $\Lambda = 1$ throughout. The notation is as in the body of the paper, *i.e.*, the PF equations read

$$\mathcal{L}_i \Pi_{SW} = 0, \quad i = 1, 2, \dots, r$$

Wherever applicable, our results are coincident with those in the literature [11, 22, 25, 26].

- $N_f = 0$ $SU(3)$

$$y^2 = (x^3 + ux + t)^2 - 1$$

$$\begin{aligned} \mathcal{L}_1 &= (27 + 4u^3 - 27t^2) \frac{\partial^2}{\partial u^2} + 12u^2 t \frac{\partial^2}{\partial u \partial t} + 3tu \frac{\partial}{\partial t} + u \\ \mathcal{L}_2 &= (27 + 4u^3 - 27t^2) \frac{\partial^2}{\partial t^2} - 36ut \frac{\partial^2}{\partial u \partial t} - 9t \frac{\partial}{\partial t} - 3 \end{aligned}$$

- $N_f = 1$ $SU(3)$

$$y^2 = (x^3 + ux + t)^2 - x$$

$$\begin{aligned} \mathcal{L}_1 &= \left(-3t + \frac{16u^3}{45t} \right) \frac{\partial}{\partial t} \\ &+ \left(-9t^2 - \frac{5u^2}{9t} + \frac{28u^3}{15} \right) \frac{\partial^2}{\partial t^2} + \left(\frac{25}{4} - 12tu + \frac{16u^4}{45t} \right) \frac{\partial^2}{\partial u \partial t} - 1 \\ \mathcal{L}_2 &= \frac{1}{(336ut - 100)} \left\{ [300t - 432t^2u] \frac{\partial}{\partial t} + [-625 + 3300tu - 3456t^2u^2] \frac{\partial^2}{\partial t \partial u} \right. \\ &+ \left. [6480t^3 + 400u^2 - 1344tu^3] \frac{\partial^2}{\partial u^2} \right\} - 1 \end{aligned}$$

- $N_f = 2 \ SU(3)$

$$y^2 = (x^3 + ux + t)^2 - x^2$$

$$\begin{aligned}\mathcal{L}_1 &= \left(-3t - \frac{8u}{9t} + \frac{8u^3}{9t}\right) \frac{\partial}{\partial t} + \left(-9t^2 - \frac{8u}{3} + \frac{8u^3}{3}\right) \frac{\partial^2}{\partial t^2} \\ &\quad + \left(\frac{8}{9t} - 12tu - \frac{16u^2}{9t} + \frac{8u^4}{9t}\right) \frac{\partial^2}{\partial t \partial u} - 1 \\ \mathcal{L}_2 &= -\left(\frac{3t}{u} + 9tu\right) \frac{\partial^2}{\partial t \partial u} + \left(4 + \frac{27t^2}{2u} - 4u^2\right) \frac{\partial^2}{\partial u^2} - 1\end{aligned}$$

- $N_f = 0 \ SO(5)$

$$y^2 = (x^4 + ux^2 + t)^2 - x^2$$

$$\begin{aligned}\mathcal{L}_1 &= 4t(u^2 + 12t) \frac{\partial^2}{\partial t^2} - (27 - 48tu + 4u^3) \frac{\partial^2}{\partial t \partial u} + 24t \frac{\partial}{\partial t} + 3 \\ \mathcal{L}_2 &= (9 - 32tu) \frac{\partial^2}{\partial t \partial u} - 4(12t + u^2) \frac{\partial^2}{\partial u^2} - 8t \frac{\partial}{\partial t} - 1\end{aligned}$$

- $N_f = 1 \ SO(5)$

$$y^2 = (x^4 + ux^2 + t)^2 - x^4$$

$$\begin{aligned}\mathcal{L}_1 &= -16tu \frac{\partial^2}{\partial t \partial u} + (4 - 16t - 4u^2) \frac{\partial^2}{\partial u^2} - 1 \\ \mathcal{L}_2 &= -16tu \frac{\partial^2}{\partial t \partial u} + (4t - 16t^2 - 4tu^2) \frac{\partial^2}{\partial t^2} + (2 - 8t - 2u^2) \frac{\partial}{\partial t} - 1\end{aligned}$$

- $N_f = 2 \ SO(5)$

$$y^2 = (x^4 + ux^2 + t)^2 - x^6$$

$$\begin{aligned}\mathcal{L}_1 &= \left(3t - \frac{32}{3}tu\right) \frac{\partial^2}{\partial u \partial t} + \left(u - 4u^2 - \frac{16t}{3}\right) \frac{\partial}{\partial u^2} + \frac{8t}{3} \frac{\partial}{\partial t} - 1 \\ \mathcal{L}_2 &= (3tu - 16t^2 - 12tu^2) \frac{\partial^2}{\partial t^2} + (3t - 16tu + u^2 - 4u^3) \frac{\partial^2}{\partial t \partial u} + (2u - 8t - 8u^2) \frac{\partial}{\partial t} - 1\end{aligned}$$

- $N_f = 1$ $Sp(6)$

$$y^2 = (x^7 + ux^5 + sx^3 + tx)^2 - 1$$

$$\begin{aligned}
\mathcal{L}_1 &= -\left[24t - \frac{32}{7}su + \frac{40}{49}u^3\right] \frac{\partial}{\partial t} - \left[8s - \frac{16}{7}u^2\right] \frac{\partial}{\partial s} \\
&+ \left[-\frac{147s}{t} - 36t^2 + \frac{16}{7}stu - \frac{20}{49}tu^3\right] \frac{\partial^2}{\partial t^2} \\
&+ \left[-48st + \frac{48}{7}s^2u - \frac{245u}{t} + \frac{4}{7}tu^2 - \frac{60}{49}su^3\right] \frac{\partial^2}{\partial s \partial t} \\
&+ \left[-16s^2 - \frac{343}{t} - 24tu + \frac{92}{7}su^2 - \frac{100}{49}u^4\right] \frac{\partial^2}{\partial u \partial t} - 1 \\
\mathcal{L}_2 &= \left[\frac{294s}{t^2} + 48t\right] \frac{\partial}{\partial t} + \left[-8s + \frac{16}{7}u^2\right] \frac{\partial}{\partial s} \\
&+ \left[\frac{441s^2}{t^2} + 60st - \frac{245u}{t} + \frac{4}{7}tu^2\right] \frac{\partial^2}{\partial t \partial s} \\
&+ \left[-16s^2 - \frac{343}{t} + \frac{735}{t^2}su + 156tu + \frac{12}{7}su^2\right] \frac{\partial^2}{\partial s^2} \\
&+ \left[\frac{1029s}{t^2} + 252t - 16su + \frac{20}{7}u^3\right] \frac{\partial^2}{\partial u \partial s} - 1 \\
\mathcal{L}_3 &= \left[\frac{294s}{t^2} + 48t\right] \frac{\partial}{\partial t} + \left[-248s - \frac{1764}{t^3}s^2 + \frac{980}{t^2}u\right] \frac{\partial}{\partial s} \\
&+ \left[-196s^2 - \frac{1323s^3}{t^3} - \frac{343}{t} + \frac{1470su}{t^2} + 156tu\right] \frac{\partial^2}{\partial t \partial u} \\
&+ \left[\frac{1029s}{t^2} + 252t - 316su - \frac{2205s^2u}{t^3} + \frac{1225u^2}{t^2}\right] \frac{\partial^2}{\partial s \partial u} \\
&+ \left[-420s - \frac{3087}{t^3}s^2 + \frac{1715}{t^2}u - 4u^2\right] \frac{\partial^2}{\partial u^2} - 1
\end{aligned}$$

- $N_f = 0$ $SO(7)$

$$y^2 = (x^6 + ux^4 + sx^2 + t)^2 - x^2$$

$$\begin{aligned}
\mathcal{L}_1 &= (-180t - 268su + \frac{75u}{t}) \frac{\partial^2}{\partial u \partial s} + (-100s^2 + \frac{25s}{t} - 132tu) \frac{\partial^2}{\partial u \partial t} \\
&+ (-420s + \frac{125}{t} - 4u^2) \frac{\partial^2}{\partial u^2} - 24t \frac{\partial}{\partial t} + (-176s + \frac{50}{t}) \frac{\partial}{\partial s} - 1 \\
\mathcal{L}_2 &= (25 - 48st + \frac{16}{5}s^2u - \frac{4}{5}tu^2 - \frac{12}{25}su^3) \frac{\partial^2}{\partial t \partial s} + (-36t^2 - \frac{16}{5}stu + \frac{12}{25}tu^3) \frac{\partial^2}{\partial t^2} - 1 \\
&+ (-16s^2 - 24tu + \frac{52}{5}su^2 - \frac{36}{25}u^4) \frac{\partial^2}{\partial t \partial u} - 24t \frac{\partial}{\partial t} + (\frac{8}{5}u^2 - 8s) \frac{\partial}{\partial s} - 1 \\
\mathcal{L}_3 &= (-16s^2 - 132tu + \frac{4}{5}su^2) \frac{\partial^2}{\partial s^2} + (25 - 84st - \frac{4}{5}tu^2) \frac{\partial^2}{\partial s \partial t} \\
&+ (-180t - 16su + \frac{12}{5}u^3) \frac{\partial^2}{\partial s \partial u} - 24t \frac{\partial}{\partial t} + (\frac{8}{5}u^2 - 8s) \frac{\partial}{\partial s} - 1
\end{aligned}$$

- $N_f = 1$ $SO(7)$

$$y^2 = (x^6 + ux^4 + sx^2 + t)^2 - x^4$$

$$\begin{aligned}
\mathcal{L}_1 &= -(72t + 64su) \frac{\partial^2}{\partial u \partial s} + (16 - 16s^2 - 60tu) \frac{\partial^2}{\partial u \partial t} \\
&\quad - (96s + 4u^2) \frac{\partial^2}{\partial u^2} - 6t \frac{\partial}{\partial t} - 32s \frac{\partial}{\partial s} - 1 \\
\mathcal{L}_2 &= -(48st + 2tu^2) \frac{\partial^2}{\partial t \partial s} + (-36t^2 - 8stu + tu^3) \frac{\partial^2}{\partial t^2} \\
&\quad + (16 - 16s^2 - 24tu + 8su^2 - u^4) \frac{\partial^2}{\partial t \partial u} + (-24t - 4su + \frac{u^3}{2}) \frac{\partial}{\partial t} + (u^2 - 8s) \frac{\partial}{\partial s} - 1 \\
\mathcal{L}_3 &= (16 - 16s^2 - 60tu) \frac{\partial^2}{\partial s^2} - (48st + 2tu^2) \frac{\partial^2}{\partial s \partial t} \\
&\quad + (-72t - 16su + 2u^3) \frac{\partial^2}{\partial s \partial u} - 6t \frac{\partial}{\partial t} + (u^2 - 8s) \frac{\partial}{\partial s} - 1
\end{aligned}$$

- $N_f = 2$ $SO(7)$

$$y^2 = (x^6 + ux^4 + sx^2 + t)^2 - x^6$$

$$\begin{aligned}
\mathcal{L}_1 &= \frac{2}{9}(27 - 108t - 48su + 4u^3) \frac{\partial}{\partial t} - 8s \frac{\partial}{\partial s} + (9t - 36t^2 - 16stu + \frac{4}{3}tu^3) \frac{\partial^2}{\partial t^2} \\
&\quad + (3s - 48st - \frac{16}{3}s^2u - 4tu^2 + \frac{4}{9}su^3) \frac{\partial^2}{\partial t \partial s} \\
&\quad + (-16s^2 - 3u - 24tu + 4su^2 - \frac{4}{9}u^4) \frac{\partial^2}{\partial u \partial t} - 1 \\
\mathcal{L}_2 &= -8s \frac{\partial}{\partial s} - (36st + 4tu^2) \frac{\partial^2}{\partial t \partial s} \\
&\quad - (16s^2 + 36tu + \frac{4}{3}su^2) \frac{\partial^2}{\partial s^2} + (9 - 36t - 16su + \frac{4}{3}u^3) \frac{\partial^2}{\partial u \partial s} - 1 \\
\mathcal{L}_3 &= -8s \frac{\partial}{\partial s} - (4s^2 + 36tu) \frac{\partial^2}{\partial t \partial u} \\
&\quad + (9 - 36t - 28su) \frac{\partial^2}{\partial u \partial s} - (36s + 4u^2) \frac{\partial^2}{\partial u^2} - 1
\end{aligned}$$

- $N_f = 0$ $SU(4)$

$$y^2 = (x^4 + sx^2 + ux + t)^2 - 1$$

$$\begin{aligned}
\mathcal{L}_1 &= (s^2 - 8t) \frac{\partial}{\partial t} - 3u \frac{\partial}{\partial u} + (16 - 16t^2 + 3su^2) \frac{\partial^2}{\partial t^2} \\
&+ (7s^2u - 24tu) \frac{\partial^2}{\partial u \partial t} + (2s^3 - 16st - 9u^2) \frac{\partial^2}{\partial s \partial t} - 1 \\
\mathcal{L}_2 &= (s^2 - 8t) \frac{\partial}{\partial t} - 3u \frac{\partial}{\partial u} + \left(-\frac{32s}{u} + \frac{32st^2}{u} + s^2u - 24tu \right) \frac{\partial^2}{\partial t \partial u} \\
&+ (2s^3 - 16st - 9u^2) \frac{\partial^2}{\partial u^2} + \left(-\frac{64}{u} + \frac{64t^2}{u} - 12su \right) \frac{\partial^2}{\partial s \partial u} - 1 \\
\mathcal{L}_3 &= \left(16t + \frac{32s}{u^2} - \frac{32st^2}{u^2} \right) \frac{\partial}{\partial t} - 3u \frac{\partial}{\partial u} + \left(32st + \frac{64s^2}{u^2} - \frac{64s^2t^2}{u^2} - 9u^2 \right) \frac{\partial^2}{\partial t \partial s} \\
&+ \left(-\frac{64}{u} + \frac{64t^2}{u} - 12su \right) \frac{\partial^2}{\partial u \partial s} + \left(-4s^2 + 96t + \frac{128s}{u^2} - \frac{128st^2}{u^2} \right) \frac{\partial^2}{\partial s^2} - 1
\end{aligned}$$

- $N_f = 1$ $SU(4)$

$$y^2 = (x^4 + sx^2 + ux + t)^2 - x$$

$$\begin{aligned}
\mathcal{L}_1 &= \frac{1}{(s^2 + 28t)} \left\{ \left[\left(\frac{4}{7}s^4 + 8s^2t - 224t^2 + 27su^2 \right) \frac{\partial}{\partial t} + (9s^2u - 84tu) \frac{\partial}{\partial u} \right] \right. \\
&+ \left[-\frac{4}{7}s^4t - 32s^2t^2 - 448t^3 - \frac{147}{4}su + 120stu^2 \right] \frac{\partial^2}{\partial t^2} \\
&+ \left[\frac{49}{4}s^2 + 343t + \frac{4}{7}s^4u + 184s^2tu - 672t^2u + 27su^3 \right] \frac{\partial^2}{\partial t \partial u} \\
&+ \left. \left[\frac{12}{7}s^5 + 32s^3t - 448st^2 + 27s^2u^2 - 252tu^2 \right] \frac{\partial^2}{\partial s \partial t} \right\} - 1 \\
\mathcal{L}_2 &= \frac{1}{(160ut - 49)} \left\{ \left[-28s^2 + 392t + 112s^2tu - 704t^2u \right] \frac{\partial}{\partial t} + \left[\frac{64}{7}s^3t + 256st^2 + 147u - 480tu^2 \right] \frac{\partial}{\partial u} \right. \\
&+ \left[\frac{-2401}{4} + \frac{1024s^3t^2}{7} + 4096st^3 - 28s^2u + 3136tu + 112s^2tu^2 - 3264t^2u^2 \right] \frac{\partial^2}{\partial t \partial u} \\
&+ \left[-84s^3 + 784st + \frac{2112}{7}s^3tu - 1792st^2u + 441u^2 - 1440tu^3 \right] \frac{\partial^2}{\partial u^2} \\
&+ \left. \left[\frac{64s^4t}{7} + 512s^2t^2 + 7168t^3 + 588su - 1920stu^2 \right] \frac{\partial^2}{\partial s \partial u} \right\} - 1 \\
\mathcal{L}_3 &= \frac{1}{(256st^2 - 49u + 196u^2t)} \left\{ \left[\frac{2401}{4} - 6144st^3 - 3185tu + 3136t^2u^2 \right] \frac{\partial}{\partial t} \right. \\
&+ \left[-196st - 128st^2u + 147u^2 - 588tu^3 \right] \frac{\partial}{\partial u} \\
&+ \left[\frac{7203}{4}s - 16384s^2t^3 - 9212stu + 6272st^2u^2 + 441u^3 - 1764tu^4 \right] \frac{\partial^2}{\partial t \partial s} \\
&+ \left[-196s^2t - 5488t^2 - 2432s^2t^2u + 17920t^3u + 588su^2 - 2352stu^3 \right] \frac{\partial^2}{\partial u \partial s} \\
&+ \left. \left[\frac{16807}{4} - 1024s^3t^2 - 28672st^3 + 196s^2u - 21952tu - 784s^2tu^2 + 22848t^2u^2 \right] \frac{\partial^2}{\partial s^2} \right\} - 1
\end{aligned}$$

- $N_f = 2 \text{ } SU(4)$

$$y^2 = (x^4 + sx^2 + ux + t)^2 - x^2$$

$$\begin{aligned}
\mathcal{L}_1 &= \frac{1}{(s^2 + 12t)} \left\{ \left[(-8s^2t - 96t^2 + 27su^2) \frac{\partial}{\partial t} + (9s^2u - 36tu) \frac{\partial}{\partial u} \right] \right. \\
&+ \left[-\frac{4}{3}s^4t - 32s^2t^2 - 192t^3 + 72stu^2 \right] \frac{\partial^2}{\partial t^2} + \left[-27su + 72s^2tu - 288t^2u + 27su^3 \right] \frac{\partial^2}{\partial t \partial u} \\
&+ \left. \left[9s^2 + \frac{4s^5}{3} + 108t - 192st^2 + 27s^2u^2 - 108tu^2 \right] \frac{\partial^2}{\partial t \partial s} \right\} - 1 \\
\mathcal{L}_2 &= \left[\frac{s^2}{2} - 2t \right] \frac{\partial}{\partial t} + \left[\frac{2s^3}{9u} + \frac{8st}{3u} - 3u \right] \frac{\partial}{\partial u} \\
&+ \left[\frac{-s^2}{2u} - \frac{6t}{u} + \frac{16s^3t}{9u} + \frac{64st^2}{3u} + \frac{s^2u}{2} - 18tu \right] \frac{\partial^2}{\partial t \partial u} + (9 + 2s^3 - 8st - 9u^2) \frac{\partial^2}{\partial u^2} \\
&+ \left[\frac{2s^4}{9u} + \frac{16s^2t}{3u} + \frac{32t^2}{u} - 12su \right] \frac{\partial^2}{\partial s \partial u} - 1 \\
\mathcal{L}_3 &= \frac{1}{(-9 + 32st + 9u^2)} \left\{ \left[(72t - 256st^2 + 144tu^2) \frac{\partial}{\partial t} + (27u - 27u^3) \frac{\partial}{\partial u} \right] \right. \\
&+ \left[-81 + 576st - 1024s^2t^2 + 162u^2 + 288stu^2 - 81u^4 \right] \frac{\partial^2}{\partial t \partial s} \\
&+ \left[108su - 288s^2tu + 1152t^2u - 108su^3 \right] \frac{\partial^2}{\partial u \partial s} \\
&+ \left. \left[36s^2 + 432t - 128s^3t - 1536st^2 - 36s^2u^2 + 1296tu^2 \right] \frac{\partial^2}{\partial s^2} \right\} - 1
\end{aligned}$$

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